

WEYL MODULES AND WEYL FUNCTORS FOR LIE SUPERALGEBRAS

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ABSTRACT. Given an algebraically closed field \mathbb{k} of characteristic zero, a Lie superalgebra \mathfrak{g} over \mathbb{k} and an associative, commutative \mathbb{k} -algebra A with unit, a Lie superalgebra of the form $\mathfrak{g} \otimes_{\mathbb{k}} A$ is known as a map superalgebra. We extend the definition of global and local Weyl modules for all map superalgebras where \mathfrak{g} is either $\mathfrak{sl}(n, n)$ with $n \geq 2$, or a finite-dimensional simple Lie superalgebra not of type $\tilde{S}(n)$ or $\mathfrak{q}(n)$. Under some mild assumptions, we prove that global and local Weyl modules satisfy certain universal, finiteness and tensor product decomposition properties. We also define Weyl functors associated to $\mathfrak{g} \otimes_{\mathbb{k}} A$ and prove that they satisfy some interesting homological properties.

CONTENTS

1. Introduction	1
2. Preliminaries	4
3. Triangular decompositions	7
4. Generalized Kac modules	14
5. Global Weyl modules	15
6. Weyl functors	17
7. The structure of $W_A(\lambda)$ as a right \mathbf{A}_λ -module	22
8. Tensor products of global Weyl modules	26
9. Local Weyl modules	28
References	30

1. INTRODUCTION

1.1. Motivation. Let \mathfrak{g} be a Lie algebra and X be a scheme, both defined over a field \mathbb{k} . Map Lie algebras (also known as generalized current Lie algebras) are Lie algebras of regular maps from X to \mathfrak{g} . They form a large class of Lie algebras generalizing loop algebras and current algebras, which are very important to the theory of affine Kac-Moody Lie algebras, and whose representation theory is an extremely active area of research.

Given a finite-dimensional, simple Lie algebra \mathfrak{g} over \mathbb{C} , (local) Weyl modules for the loop algebra $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t^{\pm 1}]$ were introduced by Chari and Pressley in [CP01]. These modules are indexed by dominant integral weights of \mathfrak{g} and are closely related to certain irreducible modules for quantum affine algebras known as quantum Weyl modules. In [FL04], Feigin and Loktev defined local and global Weyl modules for map Lie algebras of the form $\mathfrak{g} \otimes_{\mathbb{C}} A$, where \mathfrak{g} is a finite-dimensional

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semisimple Lie algebra and A is the coordinate ring of an affine variety, both defined over \mathbb{C} . A more general approach was taken in [CFK10], where Chari, Fourier and Khandai studied local Weyl modules, global Weyl modules, and Weyl functors for map algebras of the form $\mathfrak{g} \otimes_{\mathbb{C}} A$, where \mathfrak{g} is a finite-dimensional simple Lie algebra and A is an associative, commutative algebra with unit, both defined over \mathbb{C} . In [FKKS12] and [FMS15], the representation theories of local and global Weyl modules were developed for equivariant map Lie algebras, that is, Lie algebras of Γ -equivariant regular maps from an affine scheme of finite type X to a finite-dimensional simple Lie algebra \mathfrak{g} , both defined over an algebraically closed field \mathbb{k} of characteristic zero, on which a finite group Γ acts by automorphisms (both on \mathfrak{g} and X) and freely on the rational points of X .

In [CLS], Calixto, Lemay and Savage initiated the study of Weyl modules for Lie superalgebras, by defining local and global Weyl modules for map superalgebras of the form $\mathfrak{g} \otimes_{\mathbb{C}} A$, where \mathfrak{g} is either a finite-dimensional basic classical Lie superalgebra, or $\mathfrak{sl}(n, n)$ with $n \geq 2$, and A is an associative, commutative algebra with unit, both defined over \mathbb{C} .

In the current paper we study global and local Weyl modules for a more general class of map superalgebras, and initiate the study of Weyl functors in the super setting. In fact, we consider map superalgebras $\mathfrak{g} \otimes_{\mathbb{k}} A$, where \mathfrak{g} is either $\mathfrak{sl}(n, n)$ with $n \geq 2$, or any finite-dimensional simple Lie superalgebra not of type $\tilde{S}(n)$ or $\mathfrak{q}(n)$, and A is an associative, commutative algebra with unit, both defined over an algebraically closed field \mathbb{k} of characteristic zero.

1.2. Main results. Let \mathbb{k} be an algebraically closed field of characteristic zero, \mathfrak{g} be either $\mathfrak{sl}(n, n)$ with $n \geq 2$, or a finite-dimensional simple Lie superalgebra not of type $\tilde{S}(n)$ or $\mathfrak{q}(n)$, and A be an associative, commutative algebra with unit, both defined over \mathbb{k} .

In Section 3, we fix some notation and study triangular decompositions of \mathfrak{g} . Namely, we choose a reductive Lie algebra $\mathfrak{r} \subseteq \mathfrak{g}$, fix a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{r}$, and, with respect to the adjoint action of \mathfrak{h} on \mathfrak{g} , we consider certain triangular decompositions $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. In Propositions 3.1 and 3.4, we prove that \mathfrak{g} admits at least one triangular decomposition satisfying:

$$(1.1) \quad \mathfrak{n}^- \cap \mathfrak{g}_0 \subseteq \mathfrak{r}.$$

Moreover, if \mathfrak{g} is also not of type $\mathfrak{p}(n)$, then \mathfrak{g} admits at least one triangular decomposition satisfying:

$$(1.2) \quad \mathfrak{n}^- \cap \mathfrak{g}_0 \subseteq \mathfrak{r} \quad \text{and} \quad \text{the lowest root of } \mathfrak{g} \text{ is a root of } \mathfrak{r}.$$

Triangular decompositions of \mathfrak{g} satisfying (1.1) and (1.2) may be of interest on their own. We use triangular decompositions satisfying (1.1) in Section 4 to extend the definition of generalized Kac modules, which had been given by Coulembier in [Cou] only for basic classical Lie superalgebras. These generalized Kac modules are certain \mathfrak{g} -modules $K(\lambda)$, indexed by λ in a subset $X^+ \subseteq \mathfrak{h}^*$ (see (2.2) for the precise definition of X^+). We define generalized Kac modules, in Definition 4.1, via generators and relations. Then we prove in Proposition 4.2 that generalized Kac modules are finite dimensional and in Proposition 4.3 that they satisfy a certain universal property. They are thus analogues of Weyl modules for finite-dimensional simple Lie algebras, and are also used in Section 5 to define global Weyl modules.

We begin Section 5 by defining, for every \mathfrak{g} -module V , the induced module

$$P_A(V) = \mathbf{U}(\mathfrak{g} \otimes A) \otimes_{\mathbf{U}(\mathfrak{g})} V,$$

and, for every $\lambda \in X^+$ and every $\mathfrak{g} \otimes_{\mathbb{k}} A$ -module that is finitely-semisimple as an \mathfrak{r} -module, the truncated module

$$M^\lambda = M / \sum_{\mu \preceq \lambda} \mathbf{U}(\mathfrak{g} \otimes_{\mathbb{k}} A) M_\mu,$$

where M_μ , $\mu \in \mathfrak{h}^*$, denotes the weight space $\{m \in M \mid hm = \mu(h)m, \text{ for all } h \in \mathfrak{h}\}$. The global Weyl module $W_A(\lambda)$, $\lambda \in X^+$, is thus defined, in Definition 5.5, to be $P_A(K(\lambda))^\lambda$. In Proposition 5.6 we show that the global Weyl module $W_A(\lambda)$ is generated by a highest-weight vector $w_\lambda \in W_A(\lambda)_\lambda$ satisfying certain relations, and in Proposition 5.7, we prove that $W_A(\lambda)$ satisfies a certain universal property. These global Weyl modules are thus generalizations of those defined in [CFK10] and [CLS].

In Section 6, we consider commutative \mathbb{k} -algebras

$$\mathbf{A}_\lambda = \mathbf{U}(\mathfrak{h} \otimes_{\mathbb{k}} A) / \text{Ann}_{\mathfrak{h} \otimes_{\mathbb{k}} A}(w_\lambda), \quad \text{where } \text{Ann}_{\mathfrak{h} \otimes_{\mathbb{k}} A}(w_\lambda) = \{u \in \mathbf{U}(\mathfrak{h} \otimes_{\mathbb{k}} A) \mid uw_\lambda = 0\} \text{ and } \lambda \in X^+.$$

These algebras \mathbf{A}_λ are very important for the structure of global Weyl modules and will be used in Sections 6 and 7, due to the facts that $W_A(\lambda)_\lambda = \mathbf{U}(\mathfrak{h} \otimes_{\mathbb{k}} A)w_\lambda \cong \mathbf{A}_\lambda$ and $W_A(\lambda) = \mathbf{U}(\mathfrak{g} \otimes_{\mathbb{k}} A)w_\lambda$. In fact, for every $\lambda \in X^+$, since w_λ is a highest-weight generator of $W_A(\lambda)$, the global Weyl module $W_A(\lambda)$ admits a structure of (right) \mathbf{A}_λ -module (see (6.1) for the precise definition of this action). We are thus able to define, in Definition 6.3, the Weyl functor \mathbf{W}_A^λ as the functor $W_A(\lambda) \otimes_{\mathbf{A}_\lambda} -$ from the category of (left) \mathbf{A}_λ -modules to a certain full subcategory of the category of left $\mathbf{U}(\mathfrak{g} \otimes_{\mathbb{k}} A)$ -modules. In Proposition 6.5, Theorem 6.9 and Corollary 6.10, we give homological descriptions of Weyl functors. They are thus a generalization to the super setting of the Weyl functors defined in the non-super setting in [CFK10].

Assume now that \mathfrak{g} is simple and admits a triangular decomposition satisfying (1.2) (in particular, \mathfrak{g} is not isomorphic to either $\mathfrak{p}(n)$ or $\mathfrak{sl}(n, n)$), and that A is finitely generated. In this case, for each $\lambda \in X^+$, we are able to show in Theorem 7.5 that $W_A(\lambda)$ is a finitely-generated right \mathbf{A}_λ -module, in Proposition 7.6 that \mathbf{A}_λ is a finitely-generated algebra, and thus in Corollary 7.7 that $\mathbf{W}_A^\lambda V$ is a finitely-generated $\mathfrak{g} \otimes_{\mathbb{k}} A$ -module (resp. finite dimensional), if V is a finitely-generated \mathbf{A}_λ -module (resp. finite dimensional). It should be pointed out that such conditions on triangular decompositions of \mathfrak{g} are not required in the non-super setting. This is due to the fact that, unlike the non-super setting, Borel subsuperalgebras of simple Lie superalgebras are not all conjugated under the action of the Weyl group.

In Section 8, we describe the interaction between Weyl functors and tensor products. Given $\lambda, \mu \in X^+$ such that $\lambda + \mu \in X^+$, and associative, commutative, finitely-generated \mathbb{k} -algebras with unit A, B , let $C = A \oplus B$,

$$\mathbf{B}_\mu = \mathbf{U}(\mathfrak{h} \otimes_{\mathbb{k}} B) / \text{Ann}_{\mathfrak{h} \otimes_{\mathbb{k}} B}(w_\mu) \quad \text{and} \quad \mathbf{C}_{\lambda+\mu} = \mathbf{U}(\mathfrak{h} \otimes_{\mathbb{k}} C) / \text{Ann}_{\mathfrak{h} \otimes_{\mathbb{k}} C}(w_{\lambda+\mu}).$$

We show in Proposition 8.2, for every \mathbf{A}_λ -module M and \mathbf{B}_μ -module N , that $M \otimes_{\mathbb{k}} N$ admits a structure of $\mathbf{C}_{\lambda+\mu}$ -module, that $\mathbf{W}_A^\lambda M \otimes_{\mathbb{k}} \mathbf{W}_B^\mu N$ admits a structure of $\mathfrak{g} \otimes_{\mathbb{k}} C$ -module, and that as a $\mathfrak{g} \otimes_{\mathbb{k}} C$ -module, $\mathbf{W}_A^\lambda M \otimes_{\mathbb{k}} \mathbf{W}_B^\mu N$ is a quotient of $\mathbf{W}_C^{\lambda+\mu}(M \otimes_{\mathbb{k}} N)$. Moreover, we prove in Theorem 8.5 that if M, N, A, B are finite dimensional, then Weyl functors satisfy the following tensor product decomposition property:

$$\mathbf{W}_C^{\lambda+\mu}(M \otimes_{\mathbb{k}} N) \cong \mathbf{W}_A^\lambda M \otimes_{\mathbb{k}} \mathbf{W}_B^\mu N.$$

In Section 9, we define local Weyl modules in terms of generators and relations in Definition 9.1. Still assuming that \mathfrak{g} is simple, admits a triangular decomposition satisfying (1.2) and that A is finitely generated, we are able to prove in Corollary 9.5 that every local Weyl module is finite dimensional, and in Proposition 9.3 that it satisfies a certain universal property. They are thus a generalization of the local Weyl modules defined in [CFK10]. We finish the paper by characterizing local Weyl modules in terms of Weyl functors in Theorem 9.4 and by proving that they satisfy certain homological and tensor product decomposition properties in Corollary 9.7 and Theorem 9.9.

1.3. Future directions. The results of this paper open a few directions for future research. We list four directions that we consider to be the most natural ones.

- (a) Study of global Weyl modules and Weyl functors for Lie superalgebras of types $\mathfrak{q}(n)$ and $\tilde{S}(n)$. Recall that a crucial point for the definition of global Weyl modules (and thus Weyl functors) in the current paper is the definition of generalized Kac modules for the Lie superalgebras we are considering. But root space decompositions of Lie superalgebras of types $\mathfrak{q}(n)$ and $\tilde{S}(n)$ are very different from the ones that we are dealing with in this paper. Thus it is yet unclear to the authors if Lie superalgebras of types $\mathfrak{q}(n)$ and $\tilde{S}(n)$ admit a triangular decomposition satisfying (1.1), how to define generalized Kac modules, and thus how to define global Weyl modules and Weyl functors for these superalgebras.
- (b) Study of Weyl modules and Weyl functors for equivariant map superalgebras. Recall that our constructions depend on the choice of a certain reductive Lie subalgebra \mathfrak{r} of \mathfrak{g} . So, in order to study Weyl modules and Weyl functors for equivariant map superalgebras, one needs to determine which group actions on \mathfrak{g} preserve such a reductive Lie algebra, and how this group action interacts with triangular decompositions of \mathfrak{g} .
- (c) Computation of character formulas for local Weyl modules. In the non-super setting, character formulas for local Weyl modules were studied in [CP01, CFK10], using tensor product decomposition properties. In the super setting, Kac and Wakimoto conjectured several character formulas for integrable highest-weight modules for Lie superalgebras in [KW94, KW01]. Some of these formulas were proved in [Ser11] for certain affine Lie superalgebras. Since generalized Kac modules and local Weyl modules are integrable highest-weight modules and particular cases of global Weyl modules, it would be interesting to study their character formulas and see if they are related to the formulas conjectured by Kac and Wakimoto.
- (d) Relation between Weyl modules associated to different triangular decompositions. Recall that our constructions depend on the choice of a triangular decomposition of \mathfrak{g} . Since Borel sub-superalgebras of simple Lie superalgebras are not all conjugated under the action of the Weyl group, it would be interesting to describe the relation between Weyl modules associated to different triangular decompositions. For instance, relations between Kac, Verma and irreducible modules associated to different triangular decompositions of \mathfrak{g} were given in [Ser11, Cou]. Moreover, since we assume specific choices of triangular decompositions in order to obtain some of the results in Sections 7, 8 and 9, it would be interesting to verify which of these results remain valid when one chooses another triangular decomposition.

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Note on the arXiv version. For the interested reader, the tex file of the arXiv version of this paper includes hidden details of some computations, arguments and proofs that are omitted in the pdf file. These details can be displayed by switching the `details` toggle to true in the tex file and recompiling it.

2. PRELIMINARIES

Throughout this paper \mathbb{k} will denote an algebraically closed field of characteristic zero, \mathbb{Z} will denote the set of integers, $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ will denote the quotient ring $\mathbb{Z}/2\mathbb{Z}$, \mathbb{N} will denote the set $\{0, 1, \dots\}$, and \mathbb{N}_+ will denote the set $\{1, 2, \dots\}$. All vector spaces, algebras, and tensor products will be considered over the field \mathbb{k} (unless otherwise specified).

A Lie superalgebra is a \mathbb{Z}_2 -graded vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with a bracket $[\cdot, \cdot] : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ which preserves the \mathbb{Z}_2 -grading and satisfies graded versions of the axioms used to define Lie algebras. Given a Lie superalgebra \mathfrak{g} , we will denote by $\mathbf{U}(\mathfrak{g})$ its *universal enveloping superalgebra*. Recall that the superalgebra $\mathbf{U}(\mathfrak{g})$ admits a PBW-type basis, that is, if x_1, \dots, x_m is a basis of \mathfrak{g}_0 and y_1, \dots, y_n is a basis of \mathfrak{g}_1 , then the monomials

$$y_1^{j_1} \dots y_n^{j_n} x_1^{i_1} \dots x_m^{i_m}, \quad i_1, \dots, i_m \geq 0 \text{ and } j_1, \dots, j_n \in \{0, 1\},$$

form a basis of $\mathbf{U}(\mathfrak{g})$.

Let $\mathfrak{f} = \mathfrak{f}_0 \oplus \mathfrak{f}_1$, $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be Lie superalgebras, and $M = M_0 \oplus M_1$, $N = N_0 \oplus N_1$ be \mathfrak{g} -modules. Throughout this paper we will assume that every *homomorphism of Lie superalgebras* $\phi : \mathfrak{f} \rightarrow \mathfrak{g}$ and every *homomorphism of \mathfrak{g} -modules* $\psi : M \rightarrow N$ is even, that is, $\phi(\mathfrak{f}_0) \subseteq \mathfrak{g}_0$, $\phi(\mathfrak{f}_1) \subseteq \mathfrak{g}_1$, $\psi(M_0) \subseteq N_0$, and $\psi(M_1) \subseteq N_1$. Notice that the category of \mathfrak{g} -modules is equivalent to the category of left \mathbb{Z}_2 -graded $\mathbf{U}(\mathfrak{g})$ -modules. In particular, the universal enveloping superalgebra $\mathbf{U}(\mathfrak{g})$ is a \mathfrak{g} -module via left multiplication.

Given a Lie superalgebra \mathfrak{g} , a Lie subsuperalgebra $\mathfrak{t} \subseteq \mathfrak{g}$ and a \mathfrak{t} -module M , define the induced module $\text{ind}_{\mathfrak{t}}^{\mathfrak{g}} M$ to be the \mathfrak{g} -module

$$\text{ind}_{\mathfrak{t}}^{\mathfrak{g}} M = \mathbf{U}(\mathfrak{g}) \otimes_{\mathbf{U}(\mathfrak{t})} M,$$

with action induced by left multiplication. Next, we recall four standard results whose proofs will be omitted (see, for instance, [Kum02, Section 3.1 and Appendix D]).

Lemma 2.1. *Let \mathfrak{g} be a Lie superalgebra and $\mathfrak{s} \subseteq \mathfrak{t} \subseteq \mathfrak{g}$ be Lie subsuperalgebras. For every \mathfrak{s} -module M , there exists an isomorphism of \mathfrak{g} -modules*

$$\text{ind}_{\mathfrak{s}}^{\mathfrak{g}} M \cong \text{ind}_{\mathfrak{t}}^{\mathfrak{g}} \text{ind}_{\mathfrak{s}}^{\mathfrak{t}} M$$

that is functorial in M .

Lemma 2.2 (Frobenius Reciprocity). *Let \mathfrak{g} be a Lie superalgebra and $\mathfrak{t} \subseteq \mathfrak{g}$ be a Lie subsuperalgebra. For every \mathfrak{g} -module M and \mathfrak{t} -module N , there exists an isomorphism of vector spaces*

$$\text{Hom}_{\mathfrak{g}}(\text{ind}_{\mathfrak{t}}^{\mathfrak{g}} N, M) \cong \text{Hom}_{\mathfrak{t}}(N, M)$$

that is functorial in N and M .

Corollary 2.3. *Let \mathfrak{g} be a Lie superalgebra and $\mathfrak{t} \subseteq \mathfrak{g}$ be a Lie subsuperalgebra. If M is a projective \mathfrak{t} -module, then $\text{ind}_{\mathfrak{t}}^{\mathfrak{g}} M$ is a projective \mathfrak{g} -module.*

Definition 2.4 (Finitely-semisimple module). Let \mathfrak{g} be a Lie superalgebra. A \mathfrak{g} -module M is said to be *finitely semisimple* if it is equal to the direct sum of its finite-dimensional irreducible submodules. Given a subsuperalgebra $\mathfrak{t} \subseteq \mathfrak{g}$, let $\mathcal{C}_{(\mathfrak{g}, \mathfrak{t})}$ denote the full subcategory of the category of all \mathfrak{g} -modules whose objects are the \mathfrak{g} -modules which are finitely semisimple as \mathfrak{t} -modules.

Lemma 2.5. *Category $\mathcal{C}_{(\mathfrak{g}, \mathfrak{t})}$ is closed under taking submodules, quotients, arbitrary direct sums, and finite tensor products.*

Since the category of \mathfrak{g} -modules is abelian, Lemma 2.5 implies that $\mathcal{C}_{(\mathfrak{g}, \mathfrak{t})}$ is also an abelian category.

Given a Lie superalgebra \mathfrak{g} , a Lie subsuperalgebra $\mathfrak{t} \subseteq \mathfrak{g}$, and a \mathfrak{t} -module M , notice that, for every $u \otimes m \in \text{ind}_{\mathfrak{t}}^{\mathfrak{g}} M$ and every homogeneous element $x \in \mathfrak{t}$, we have

$$x \cdot (u \otimes m) = xu \otimes m = ([x, u] + (-1)^{p(u)p(x)}ux) \otimes m = [x, u] \otimes m + (-1)^{p(u)p(x)}u \otimes x \cdot m.$$

\mathfrak{g}	\mathfrak{t}	Type
$A(m, n), m > n \geq 0$	$A_m \oplus A_n \oplus \mathbb{k}$	Basic, type I
$A(n, n), n \geq 1$	$A_n \oplus A_n$	Basic, type I
$\mathfrak{sl}(n, n), n \geq 2$	$A_{n-1} \oplus A_{n-1} \oplus \mathbb{k}$	N/A
$B(m, n), m \geq 0, n \geq 1$	$B_m \oplus C_n$	Basic, type II
$C(n+1), n \geq 1$	$C_n \oplus \mathbb{k}$	Basic, type I
$D(m, n), m \geq 2, n \geq 1$	$D_m \oplus C_n$	Basic, type II
$D(2, 1; \alpha), \alpha \neq 0, -1$	$A_1 \oplus A_1 \oplus A_1$	Basic, type II
$F(4)$	$A_1 \oplus B_3$	Basic, type II
$G(3)$	$A_1 \oplus G_2$	Basic, type II
$H(n), n \geq 4$	B_n or D_n	Cartan
$S(n), n \geq 3$	A_{n-1}	Cartan
$\tilde{S}(n), n = 2m, m \geq 2$	A_{n-1}	Cartan
$W(n), n \geq 2$	$A_{n-1} \oplus \mathbb{k}$	Cartan
$\mathfrak{p}(n), n \geq 2$	A_n	Strange
$\mathfrak{q}(n), n \geq 2$	A_n	Strange

TABLE 1.

Recall that $\mathbf{U}(\mathfrak{g})$ can also be considered as a \mathfrak{t} -module via the restriction of the adjoint representation. Thus, as a \mathfrak{t} -module, $\text{ind}_{\mathfrak{t}}^{\mathfrak{g}} M$ is the tensor product of $\mathbf{U}(\mathfrak{g})$ (considered as a \mathfrak{t} -module via the restriction of the adjoint representation) and M .

Lemma 2.6. *Let \mathfrak{g} be a Lie superalgebra, $\mathfrak{t} \subseteq \mathfrak{g}$ be a Lie subsuperalgebra and M be a \mathfrak{t} -module. If \mathfrak{g} (via the adjoint representation) and M are finitely-semisimple \mathfrak{t} -modules, then $\text{ind}_{\mathfrak{t}}^{\mathfrak{g}} M$ is an object in $\mathcal{C}_{(\mathfrak{g}, \mathfrak{t})}$.*

Proof. Since \mathfrak{g} is assumed to be a finitely-semisimple \mathfrak{t} -module via the adjoint representation, by Lemma 2.5, \mathfrak{g} , its tensor algebra, and $\mathbf{U}(\mathfrak{g})$ are finitely-semisimple \mathfrak{t} -modules. Since M is assumed to be a finitely-semisimple \mathfrak{t} -module and $\text{ind}_{\mathfrak{t}}^{\mathfrak{g}} M$ is the tensor product of $\mathbf{U}(\mathfrak{g})$ (considered as a \mathfrak{t} -module via the restriction of the adjoint representation) and M , from Lemma 2.5, we conclude that $\text{ind}_{\mathfrak{t}}^{\mathfrak{g}} M$ is also a finitely-semisimple \mathfrak{t} -module. \square

Lemma 2.7. *Let \mathfrak{g} be a Lie superalgebra, $\mathfrak{t} \subseteq \mathfrak{g}$ be a Lie subsuperalgebra. If M is a cyclic \mathfrak{t} -module given as the quotient of $\mathbf{U}(\mathfrak{t})$ by a left ideal $J \subsetneq \mathbf{U}(\mathfrak{t})$, then $\text{ind}_{\mathfrak{t}}^{\mathfrak{g}} M$ is a cyclic \mathfrak{g} -module given as the quotient of $\mathbf{U}(\mathfrak{g})$ by the left ideal generated by J in $\mathbf{U}(\mathfrak{g})$.*

Proof. Using the short exact sequence $0 \rightarrow J \rightarrow \mathbf{U}(\mathfrak{t}) \rightarrow M \rightarrow 0$, this proof is straightforward. \square

Finite-dimensional simple Lie superalgebras over an algebraically closed field of characteristic zero were classified by V. Kac in [Kac77], and they can be divided into three groups: *basic classical*, *strange*, and *Cartan type* (see Table 1). Let \mathfrak{g} be either $\mathfrak{sl}(n, n)$ with $n \geq 2$, or a finite-dimensional simple Lie superalgebra not of type $\tilde{S}(n)$ or $\mathfrak{q}(n)$. These Lie superalgebras admit a \mathbb{Z} -grading $\mathfrak{g} = \bigoplus_{i \geq -2} \mathfrak{g}_i$ (see Section 3 for more details). Let

$$(2.1) \quad \mathfrak{t} = \begin{cases} \mathfrak{g}_0, & \text{if } \mathfrak{g} \text{ is of Cartan type,} \\ \mathfrak{g}_{\bar{0}}, & \text{otherwise.} \end{cases}$$

For every \mathfrak{g} , the Lie subsuperalgebra \mathfrak{r} is a reductive Lie algebra (see Table 1). Denote by \mathfrak{r}' the semisimple part of \mathfrak{r} and by \mathfrak{z} its center. Fix a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{r}$ (in particular \mathfrak{h} is abelian) and consider a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. Notice that a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ induces a triangular decomposition $\mathfrak{r} = \mathfrak{n}_0^- \oplus \mathfrak{h} \oplus \mathfrak{n}_0^+$, where $\mathfrak{n}_0^\pm = \mathfrak{n}^\pm \cap \mathfrak{r} = \mathfrak{n}^\pm \cap \mathfrak{r}'$ and $\mathfrak{z} \subseteq \mathfrak{h}$.

A \mathfrak{g} -module V is said to be a *weight module* (with respect to \mathfrak{h}) when

$$V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu, \quad \text{where } V_\mu = \{v \in V \mid hv = \mu(h)v \text{ for all } h \in \mathfrak{h}\}.$$

An element $\mu \in \mathfrak{h}^*$ is said to be a *weight* of V when $V_\mu \neq \{0\}$, and, in this case, V_μ is said to be a *weight space* of V , and the elements of V_μ are said to be *weight vectors*. A vector $v \in V_\mu \setminus \{0\}$ is said to be a *highest-weight vector* (with respect to the fixed triangular decomposition) if $\mathfrak{n}^+v = 0$. Similarly, $\lambda \in \mathfrak{h}^*$ is said to be the *lowest weight* of a weight \mathfrak{g} -module V if $V_\lambda \neq \{0\}$ and $\mathfrak{n}^-V_\lambda = \{0\}$. A \mathfrak{g} -module V is said to be a *highest-weight module* of highest weight $\lambda \in \mathfrak{h}^*$ if V is generated by a highest-weight vector $v \in V_\lambda \setminus \{0\}$. Every irreducible finite-dimensional \mathfrak{g} -module is a highest-weight module. Denote by $L_{\mathfrak{b}}(\lambda)$ the unique *irreducible* \mathfrak{g} -module of highest weight λ (with respect to \mathfrak{b}), and set

$$(2.2) \quad X^+ = \{\lambda \in \mathfrak{h}^* \mid L_{\mathfrak{b}}(\lambda) \text{ is finite dimensional}\}.$$

3. TRIANGULAR DECOMPOSITIONS

If \mathfrak{g} is a finite-dimensional simple Lie superalgebra, with respect to each choice of triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, we have that, as a \mathfrak{g} -module, \mathfrak{g} has a lowest weight, which we will denote by $-\theta$. Let \mathfrak{n}_0^\pm denote $\mathfrak{n}^\pm \cap \mathfrak{g}_0$. In this paper we will be interested in triangular decompositions $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ satisfying the following conditions:

- (C1) $\mathfrak{n}_0^- \subseteq \mathfrak{r}$.
- (C2) $\mathfrak{n}_0^- \subseteq \mathfrak{r}$ and $-\theta$ is also a root of \mathfrak{r} .

This section is devoted to constructing triangular decompositions satisfying these conditions.

3.1. Basic classical Lie superalgebras and $\mathfrak{sl}(n, n)$ with $n \geq 2$. In this subsection we assume that \mathfrak{g} is a basic classical Lie superalgebra, unless otherwise specified. In these cases, \mathfrak{g}_0 is a reductive Lie algebra. A basic classical Lie superalgebra \mathfrak{g} is said to be of *type II* if \mathfrak{g}_1 is an irreducible \mathfrak{g}_0 -module, and \mathfrak{g} is said to be of *type I* if \mathfrak{g}_1 is a direct sum of two irreducible \mathfrak{g}_0 -modules (see Table 1).

A *Cartan subalgebra* $\mathfrak{h} \subseteq \mathfrak{g}$ is defined to be a Cartan subalgebra of \mathfrak{g}_0 . Under the adjoint action of \mathfrak{h} , we have a root space decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^* \setminus \{0\}} \mathfrak{g}_\alpha, \quad \text{where } \mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}.$$

Denote by R the *set of roots*, $\{\alpha \in \mathfrak{h}^* \setminus \{0\} \mid \mathfrak{g}_\alpha \neq \{0\}\}$. For $\alpha \in R$, \mathfrak{g}_α is either purely even, that is, $\mathfrak{g}_\alpha \subseteq \mathfrak{g}_0$, or \mathfrak{g}_α is purely odd, that is, $\mathfrak{g}_\alpha \subseteq \mathfrak{g}_1$. Let $R_0 = \{\alpha \in R \mid \mathfrak{g}_\alpha \subseteq \mathfrak{g}_0\}$ be the set of even roots and $R_1 = \{\alpha \in R \mid \mathfrak{g}_\alpha \subseteq \mathfrak{g}_1\}$ be the set of odd roots.

It is known that \mathfrak{g} can be realized as a contragredient Lie superalgebra (for details, see [Mus12, Chapter 5]). In particular, every choice of a set of simple roots $\Delta \subseteq R$ yields a decomposition

$R = R^+(\Delta) \sqcup R^-(\Delta)$, where $R^+(\Delta)$ (resp. $R^-(\Delta)$) denotes the set of positive (resp. negative) roots (defined in the usual way). Define

$$\Delta_{\bar{0}} = \Delta \cap R_{\bar{0}}, \quad \Delta_{\bar{1}} = \Delta \cap R_{\bar{1}}, \quad R_{\bar{0}}^{\pm} = R_{\bar{0}} \cap R^{\pm} \quad \text{and} \quad R_{\bar{1}}^{\pm} = R_{\bar{1}} \cap R^{\pm}.$$

A choice of simple roots $\Delta \subseteq R$ also induces a triangular decomposition $\mathfrak{g} = \mathfrak{n}^-(\Delta) \oplus \mathfrak{h} \oplus \mathfrak{n}^+(\Delta)$, where $\mathfrak{n}^{\pm}(\Delta) = \bigoplus_{\alpha \in R^{\pm}(\Delta)} \mathfrak{g}_{\alpha}$. The subsuperalgebra $\mathfrak{b}(\Delta) = \mathfrak{h} \oplus \mathfrak{n}^+(\Delta)$ is said to be the *Borel subsuperalgebra* of \mathfrak{g} corresponding to Δ .

In order to construct a triangular decomposition satisfying (C2), for each simple odd isotropic root (that is, $\beta \in \Delta_{\bar{1}}$ such that $\beta(h_{\beta}) = 0$), define the *odd reflection* with respect to β to be the map

$$r_{\beta}: \Delta \rightarrow R, \quad r_{\beta}(\beta') = \begin{cases} -\beta, & \text{if } \beta' = \beta, \\ \beta', & \text{if } \beta' \in \Delta, \beta' \neq \beta, \text{ and } \beta(h_{\beta'}) = \beta'(h_{\beta}) = 0, \\ \beta + \beta', & \text{if } \beta' \in \Delta, \beta' \neq \beta, \beta(h_{\beta'}) \neq 0 \text{ or } \beta'(h_{\beta}) \neq 0. \end{cases}$$

By [CW12, Lemma 1.30], the set $r_{\beta}(\Delta)$ is a set of simple roots in R , and

$$R^+(r_{\beta}(\Delta)) \setminus \{-\beta\} = R^+(\Delta) \setminus \{\beta\}.$$

Now, let $\Delta_{\text{dis}} = \{\gamma_1, \dots, \gamma_n\}$ be a distinguished set of simple roots of \mathfrak{g} (that is, a set of simple roots that has only one odd root), and let γ_s denote the unique odd root in Δ_{dis} (see [FSS00, Tables 3.54, 3.57-3.60]). The choice of Δ_{dis} induces a \mathbb{Z} -grading $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ that is compatible with the \mathbb{Z}_2 -grading. Namely:

$$(3.1) \quad \mathfrak{g}_{\bar{0}} = \mathfrak{g}_0 \quad \text{and} \quad \mathfrak{g}_{\bar{1}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1, \quad \text{if } \mathfrak{g} \text{ is of type I,}$$

$$(3.2) \quad \mathfrak{g}_{\bar{0}} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2 \quad \text{and} \quad \mathfrak{g}_{\bar{1}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1, \quad \text{if } \mathfrak{g} \text{ is of type II.}$$

Moreover, Δ_{dis} induces a triangular decomposition

$$(3.3) \quad \mathfrak{g} = \mathfrak{n}^-(\Delta_{\text{dis}}) \oplus \mathfrak{h} \oplus \mathfrak{n}^+(\Delta_{\text{dis}}), \quad \text{where} \quad \mathfrak{n}^{\pm}(\Delta_{\text{dis}}) = \mathfrak{n}_0^{\pm} \oplus \left(\bigoplus_{i>0} \mathfrak{g}_{\pm i} \right).$$

A subsuperalgebra $\mathfrak{b}_{\text{dis}} = \mathfrak{b}(\Delta_{\text{dis}})$ is called a distinguished Borel subsuperalgebra of \mathfrak{g} .

Recall that $A(n, n) = \mathfrak{sl}(n, n)/\mathbb{K}I_{n, n}$, where $I_{n, n}$ is the identity matrix in $\mathfrak{sl}(n, n)$. Hence, the preimage of the canonical projection $\mathfrak{sl}(n, n) \twoheadrightarrow A(n, n)$ induces decompositions as in (3.1) and (3.3) on $\mathfrak{sl}(n, n)$.

Proposition 3.1. *Let \mathfrak{g} be either a basic classical Lie superalgebra, or $\mathfrak{sl}(n, n)$ with $n \geq 2$, and let Δ_{dis} be a distinguished system of simple roots for \mathfrak{g} .*

- (a) *If \mathfrak{g} is a basic classical Lie superalgebra of type II, then the triangular decomposition of \mathfrak{g} induced by Δ_{dis} satisfies (C2).*
- (b) *If \mathfrak{g} is $\mathfrak{sl}(n, n)$, $n \geq 2$, or a basic classical Lie superalgebra of type I, then the triangular decomposition of \mathfrak{g} induced by $r_{\gamma_s}(\Delta_{\text{dis}})$ satisfies (C2).*

In particular, \mathfrak{g} admits at least one triangular decomposition satisfying (C2).

Proof. Notice that, since \mathfrak{r} is defined to be $\mathfrak{g}_{\bar{0}}$, we have $\mathfrak{n}_0^- = \mathfrak{n}^- \cap \mathfrak{g}_{\bar{0}} \subseteq \mathfrak{r}$ for every triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. That is, we only have to prove that there exists a triangular decomposition such that the associated lowest root of \mathfrak{g} is also a root of \mathfrak{r} .

(a) Recall from (3.1) that, if \mathfrak{g} is of type II, the \mathbb{Z} -grading associated to Δ_{dis} is given by $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$. Now notice that $\mathfrak{g}_{-\theta} \subseteq \mathfrak{g}_{-2}$. Since $\mathfrak{g}_{-2} \subseteq \mathfrak{g}_{\bar{0}} = \mathfrak{r}$, we obtain that $-\theta$ is also a root of \mathfrak{r} .

(b) Let λ denote the highest root of \mathfrak{g} with respect to Δ_{dis} , and let Δ denote $r_{\gamma_s}(\Delta_{\text{dis}})$. One can check that $\lambda(h_{\gamma_s}) \neq 0$. (In fact, $\lambda(h_{\gamma_s})$ is the sum of the elements of the s -th column of the Cartan matrix of \mathfrak{g} .) Thus, it follows from [Ser11, Lemma 10.2] that $L_{\mathfrak{b}_{\text{dis}}}(\lambda) \cong L_{\mathfrak{b}(\Delta)}(\lambda - \gamma_s)$. That is, $\theta = \lambda - \gamma_s$ is the highest root of \mathfrak{g} relative to $\mathfrak{b}(\Delta)$. Since both λ and γ_s are odd roots, θ is an even root. Thus, since \mathfrak{r} is defined to be $\mathfrak{g}_{\bar{0}}$, the lowest root $-\theta$ is also a root of \mathfrak{r} . \square

3.2. Cartan type Lie superalgebras. In this subsection \mathfrak{g} will denote a Lie superalgebra of Cartan type that is not isomorphic to $\tilde{S}(n)$ (see Table 1). We will now briefly describe each one of these Lie superalgebras.

Fix $n \geq 2$ and let $\Lambda(n)$ denote the exterior algebra with generators ξ_1, \dots, ξ_n . The algebra $\Lambda(n)$ is a 2^n -dimensional associative anticommutative algebra, which admits a $\mathbb{Z} \times \mathbb{Z}_2$ -grading by setting the degree of ξ_1, \dots, ξ_n to be $(1, \bar{1})$. Thus, with respect to the \mathbb{Z} -grading,

$$\Lambda(n) = \bigoplus_{k=0}^n \Lambda^k(n), \quad \text{where } \Lambda^k(n) = \text{span}_{\mathbb{k}} \{ \xi_{i_1} \cdots \xi_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n \}, \quad k \in \{0, \dots, n\},$$

and with respect to the \mathbb{Z}_2 -grading,

$$\Lambda(n) = \Lambda(n)_{\bar{0}} \oplus \Lambda(n)_{\bar{1}}, \quad \text{where } \Lambda(n)_{\bar{0}} = \bigoplus_{k=0}^{\lfloor n/2 \rfloor} \Lambda^{2k}(n) \quad \text{and} \quad \Lambda(n)_{\bar{1}} = \bigoplus_{k=0}^{\lfloor n/2 \rfloor} \Lambda^{2k+1}(n).$$

If $x \in \Lambda(n)_z$, $z \in \mathbb{Z}_2$, we say that x is homogeneous and define $p(x) = z$.

Given a linear map $D: \Lambda(n) \rightarrow \Lambda(n)$, define $p(D) = \bar{0}$, if D is even, and $p(D) = \bar{1}$, if D is odd. A *homogeneous superderivation* of $\Lambda(n)$ is defined to be a linear map $D: \Lambda(n) \rightarrow \Lambda(n)$ that is either even or odd, and satisfies $D(xy) = D(x)y + (-1)^{p(D)p(x)}xD(y)$ for all homogeneous $x, y \in \Lambda(n)$. A *superderivation* of $\Lambda(n)$ is a linear combination of homogeneous superderivations of $\Lambda(n)$.

Let $W(n)$ be the Lie superalgebra consisting of superderivations of $\Lambda(n)$ endowed with the unique superbracket satisfying

$$[D_1, D_2] = D_1 \circ D_2 - (-1)^{p(D_1)p(D_2)}D_2 \circ D_1,$$

for all homogeneous superderivations D_1, D_2 . The \mathbb{Z} -grading on $\Lambda(n)$ induces a \mathbb{Z} -grading:

$$W(n) = W(n)_{-1} \oplus W(n)_0 \oplus \cdots \oplus W(n)_{n-1},$$

where $W(n)_k$ consists of derivations that map ξ_1, \dots, ξ_n to $\Lambda(n)_{k+1}$.

For each $i \in \{1, \dots, n\}$, denote by ∂_i the unique superderivation of $\Lambda(n)$ satisfying $\partial_i(\xi_j) = \delta_{i,j}$ for all $j \in \{1, \dots, n\}$, and for each $x \in \Lambda(n)$, let D_x denote the superderivation $\sum_{i=1}^n \partial_i(x)\partial_i$. The subspace

$$\tilde{H}(n) = \text{span}_{\mathbb{k}} \{ D_x \mid x \in \Lambda(n) \} \subseteq W(n)$$

is a subsuperalgebra of $W(n)$ and inherits a \mathbb{Z} -grading $\tilde{H}(n) = \tilde{H}(n)_{-1} \oplus \tilde{H}(n)_0 \oplus \cdots \oplus \tilde{H}(n)_{n-2}$. The simple Lie superalgebra $H(n)$ is defined to be the

$$H(n) = [\tilde{H}(n), \tilde{H}(n)] = H(n)_{-1} \oplus H(n)_0 \oplus \cdots \oplus H(n)_{n-3}.$$

Now, let $\text{div}: W(n) \rightarrow W(n)$ be the linear transformation given by

$$\text{div}(D) = \sum_{i=1}^n (D(\xi_i)) \partial_i \quad \text{for all } D \in W(n).$$

The superalgebra $S(n)$ is the subsuperalgebra of $W(n)$ consisting of all $D \in W(n)$ such that $\text{div}(D) = 0$. The superalgebra $S(n)$ inherits a \mathbb{Z} -grading from $W(n)$:

$$S(n) = S(n)_{-1} \oplus S(n)_0 \oplus \cdots \oplus S(n)_{n-2}.$$

A crucial difference between Cartan type superalgebras and basic classical superalgebras is that for Cartan type superalgebras, \mathfrak{g}_0 is not a reductive Lie algebra. However, as was described above, if \mathfrak{g} is a Cartan type superalgebra not of type $\tilde{S}(n)$, then \mathfrak{g} admits a \mathbb{Z} -grading (compatible with the \mathbb{Z}_2 -grading) $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{n-1}$, and moreover, \mathfrak{g}_0 is a reductive Lie algebra.

A *Cartan subalgebra* \mathfrak{h} of \mathfrak{g} is defined to be a Cartan subalgebra of \mathfrak{g}_0 . Fix the Cartan subalgebras \mathfrak{h} of \mathfrak{g}_0 that have the following bases:

$$\begin{aligned} & \{\xi_k \partial_k \mid 1 \leq k \leq n\}, \quad \text{if } \mathfrak{g} \cong W(n); \\ & \{\xi_k \partial_k - \xi_{k+1} \partial_{k+1} \mid 1 \leq k \leq n-1\}, \quad \text{if } \mathfrak{g} \cong S(n); \\ & \{\xi_k \partial_k - \xi_{\lfloor n/2 \rfloor + k} \partial_{\lfloor n/2 \rfloor + k} \mid 1 \leq k \leq \lfloor n/2 \rfloor\}, \quad \text{if } \mathfrak{g} \cong H(n). \end{aligned}$$

Consider the element

$$\mathcal{E} := \sum_{i=1}^n \xi_i \partial_i \in W(n)_0,$$

and define $\bar{\mathfrak{g}} = \mathfrak{g} + \mathbb{k}\mathcal{E}$ and $\bar{\mathfrak{h}} = \mathfrak{h} + \mathbb{k}\mathcal{E}$. Notice that $\mathcal{E} \notin S(n)$ and $\mathcal{E} \notin H(n)$, thus

$$\mathfrak{g} = \bar{\mathfrak{g}}, \quad \text{if } \mathfrak{g} \cong W(n) \quad \text{and} \quad \mathfrak{g} \subsetneq \mathfrak{g} \oplus \mathbb{k}\mathcal{E} = \bar{\mathfrak{g}}, \quad \text{if } \mathfrak{g} \cong H(n), S(n).$$

Also notice that:

$$[\mathcal{E}, x] = zx, \quad \text{for all } x \in \mathfrak{g}_z \text{ and } z \in \mathbb{Z}.$$

Hence the adjoint action of $\bar{\mathfrak{h}}$ on \mathfrak{g} gives a root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \bar{\mathfrak{h}}^* \setminus \{0\}} \mathfrak{g}_\alpha, \quad \text{where } \mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \bar{\mathfrak{h}}\}.$$

Denote by R the set of roots, $\{\alpha \in \bar{\mathfrak{h}}^* \setminus \{0\} \mid \mathfrak{g}_\alpha \neq \{0\}\}$. For $\alpha \in R$, either $\mathfrak{g}_\alpha \subseteq \mathfrak{g}_0$, or $\mathfrak{g}_\alpha \subseteq \mathfrak{g}_{\bar{1}}$. Let $R_0 = \{\alpha \in R \mid \mathfrak{g}_\alpha \subseteq \mathfrak{g}_0\}$ be the set of even roots and $R_{\bar{1}} = \{\alpha \in R \mid \mathfrak{g}_\alpha \subseteq \mathfrak{g}_{\bar{1}}\}$ be the set of odd roots. Moreover, for each $\alpha \in R$, there is $z \in \mathbb{Z}$ such that $\mathfrak{g}_\alpha \subseteq \mathfrak{g}_z$. Thus, we can define the *height* of $\alpha \in R$ to be $\text{ht}(\alpha) = z$, and define R_z to be $\{\alpha \in R \mid \text{ht}(\alpha) = z\}$. Notice that

$$R = \bigcup_{z \in \mathbb{Z}} R_z, \quad R_0 = \bigcup_{z \in \mathbb{Z}} R_{2z} \quad \text{and} \quad R_{\bar{1}} = \bigcup_{z \in \mathbb{Z}} R_{2z+1}.$$

Remark 3.2. Notice that \mathcal{E} captures the height of the roots. Thus, it helps identifying the simple roots of \mathfrak{g} . The addition of \mathcal{E} to \mathfrak{h} will be used in the sequel to construct triangular decompositions satisfying (C2).

We will describe now the roots and root spaces of \mathfrak{g} . Notice that, if $\mathfrak{g} \cong W(n)$, then $\mathfrak{g}_0 \cong \mathfrak{gl}(n)$, with the basis elements $\xi_i \partial_j \in W(n)$ corresponding to the basis elements $E_{ij} \in \mathfrak{gl}(n)$. Recall that $\mathfrak{h} = \text{span}_{\mathbb{k}}\{\xi_i \partial_i \mid 1 \leq i \leq n\}$ and, for each $1 \leq i \leq n$, let ε_i be the unique linear map in \mathfrak{h}^* satisfying $\varepsilon_i(\xi_j \partial_j) = \delta_{i,j}$, for all $1 \leq j \leq n$. The set of roots of \mathfrak{g} is

$$R = \{\varepsilon_{i_1} + \cdots + \varepsilon_{i_k} - \varepsilon_j \mid 1 \leq i_1 < \cdots < i_k \leq n, 0 \leq k \leq n \text{ and } 1 \leq j \leq n\} \setminus \{0\},$$

and the corresponding root spaces are:

$$\mathfrak{g}_\alpha = \begin{cases} \mathbb{k} \xi_{i_1} \cdots \xi_{i_k} \partial_j, & \text{if } \alpha = \varepsilon_{i_1} + \cdots + \varepsilon_{i_k} - \varepsilon_j \text{ and } j \notin \{i_1, \dots, i_k\}, \\ \text{span}_{\mathbb{k}}\{\xi_{i_1} \cdots \xi_{i_k} \xi_j \partial_j \mid j \notin \{i_1, \dots, i_k\}\}, & \text{if } \alpha = \varepsilon_{i_1} + \cdots + \varepsilon_{i_k}. \end{cases}$$

If $\mathfrak{g} = S(n)$, then $\mathfrak{g}_0 \cong \mathfrak{sl}(n)$. The set of roots of $S(n)$ is the subset of the set of roots of $W(n)$ obtained from it by removing the roots $\varepsilon_1 + \cdots + \varepsilon_n - \varepsilon_j$ for all $1 \leq j \leq n$, that is,

$$R = \{\varepsilon_{i_1} + \cdots + \varepsilon_{i_k} - \varepsilon_j \mid 1 \leq i_1 < \cdots < i_k \leq n, 0 \leq k \leq n-1 \text{ and } 1 \leq j \leq n\} \setminus \{0\}.$$

The corresponding root spaces are thus:

$$\mathfrak{g}_\alpha = \begin{cases} \mathbb{k} \xi_{i_1} \cdots \xi_{i_k} \partial_j, & \text{if } \alpha = \varepsilon_{i_1} + \cdots + \varepsilon_{i_k} - \varepsilon_j \text{ and } j \notin \{i_1, \dots, i_k\}, \\ \text{span}_{\mathbb{k}}\{\xi_{i_1} \cdots \xi_{i_k} (\xi_j \partial_j - \xi_{j+1} \partial_{j+1}) \mid j, j+1 \notin \{i_1, \dots, i_k\}\}, & \text{if } \alpha = \varepsilon_{i_1} + \cdots + \varepsilon_{i_k}. \end{cases}$$

Finally, if $\mathfrak{g} = H(n)$, then $\mathfrak{g}_0 \cong \mathfrak{so}(n)$. Let $r = \lfloor n/2 \rfloor$, $\{\varepsilon_1, \dots, \varepsilon_r\}$ be the elements in the Cartan subalgebra of \mathfrak{g}_0 that identify with the standard basis of the Cartan subalgebra of $\mathfrak{so}(n)$, and $\delta \in \bar{\mathfrak{h}}^*$ be the dual of \mathcal{E} . If $n = 2r$, then the set of roots of \mathfrak{g} is given by

$$R = \{\pm \varepsilon_{i_1} \pm \cdots \pm \varepsilon_{i_k} + m\delta \mid 1 \leq i_1 < \cdots < i_k \leq r, k-2 \leq m \leq n-2, m \geq -1, k-m \in 2\mathbb{Z}\}.$$

If $n = 2r+1$, then the set of roots of \mathfrak{g} is the set

$$R = \{\pm \varepsilon_{i_1} \pm \cdots \pm \varepsilon_{i_k} + m\delta \mid 1 \leq i_1 < \cdots < i_k \leq r, k-2 \leq m \leq n-2, m \geq -1\}.$$

For each root $\alpha = d_1\varepsilon_1 + \cdots + d_r\varepsilon_r + m\delta$ with $d_i \in \{-1, 0, 1\}$, we have:

$$\mathfrak{g}_\alpha = \text{span}_{\mathbb{k}}\{D_x \mid x = \xi_1^{a_1} \cdots \xi_n^{a_n}, a_i \in \{0, 1\}, a_1 + \cdots + a_n = m+2, a_i - a_{r+i} = d_i \text{ for all } i\}.$$

Remark 3.3. Some properties that hold for roots of semisimple Lie algebras do not hold for Cartan type Lie superalgebras. For instance, a root may have multiplicity greater than 1, and R^- may be different from $-R^+$. (For more details, see [Gav14, Kac77, Sch79, Ser05].) Also, notice that the root space decomposition of \mathfrak{g} given by the action of $\bar{\mathfrak{h}}$ induces the root space decomposition of \mathfrak{g} given by the action of \mathfrak{h} . The roots with respect to \mathfrak{h} are the restrictions of the elements of R to \mathfrak{h} . Hence the set of roots of \mathfrak{g} with respect to \mathfrak{h} will be also denoted by R and their roots will be denoted by the same symbols.

An element $h \in \mathfrak{h}_{\mathbb{R}}$ is said to be *regular* if $\alpha(h) \neq 0$ for all $\alpha \in R$. Every regular element $h \in \mathfrak{h}_{\mathbb{R}}$ induces a decomposition $R = R^+(h) \sqcup R^-(h)$, where

$$R^+(h) = \{\alpha \in R \mid \alpha(h) > 0\} \quad \text{and} \quad R^-(h) = \{\alpha \in R \mid \alpha(h) < 0\}.$$

The set $R^+(h)$ (resp. $R^-(h)$) is said to be the *set of positive* (resp. *negative*) roots of \mathfrak{g} relative to h . A regular element $h \in \mathfrak{h}_{\mathbb{R}}$ also induces a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^-(h) \oplus \mathfrak{h} \oplus \mathfrak{n}^+(h), \quad \text{where} \quad \mathfrak{n}^\pm(h) = \bigoplus_{\alpha \in R^\pm(h)} \mathfrak{g}_\alpha.$$

A Lie subsuperalgebra \mathfrak{b} is a *Borel subsuperalgebra* of \mathfrak{g} if $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+(h)$ for some regular $h \in \mathfrak{h}_{\mathbb{R}}$.

Following [Ser05], a root $\alpha \in R^+$ is said to be *simple* for a Borel subsuperalgebra \mathfrak{b} if the set

$$r_\alpha(R^+) = \begin{cases} (R^+ \setminus \{\alpha\}) \cup \{-\alpha\}, & \text{if } -\alpha \in R, \\ R^+ \setminus \{\alpha\}, & \text{otherwise} \end{cases}$$

is a set of positive roots relative to some regular element $h \in \mathfrak{h}_{\mathbb{R}}$. In this case, the subsuperalgebra

$$r_\alpha(\mathfrak{b}) := \mathfrak{h} \oplus \bigoplus_{\beta \in r_\alpha(R^+)} \mathfrak{g}_\beta$$

is said to be the Borel subsuperalgebra of \mathfrak{g} obtained from \mathfrak{b} by the reflection r_α .

Choose a Borel subalgebra \mathfrak{b}_0 of \mathfrak{g}_0 such that the set of simple roots associated to it is given by:

$$\begin{aligned} &\{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n\}, & \text{if } \mathfrak{g} \cong W(n), S(n), \\ &\{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{r-1} - \varepsilon_r, \varepsilon_{r-1} + \varepsilon_r\}, & \text{if } \mathfrak{g} \cong H(2r), \\ &\{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{r-1} - \varepsilon_r, \varepsilon_r\}, & \text{if } \mathfrak{g} \cong H(2r+1). \end{aligned}$$

Let R_0^+ (resp. R_0^-) denote the set of positive (resp. negative) roots of \mathfrak{g}_0 associated to the simple roots above. The subsuperalgebra $\mathfrak{b}_{\max} = \mathfrak{b}_0 \oplus (\bigoplus_{i>1} \mathfrak{g}_i)$ is known as the *maximal Borel subsuperalgebra* of \mathfrak{g} , and $\mathfrak{b}_{\min} = \mathfrak{b}_0 \oplus \mathfrak{g}_{-1}$ is known as the *minimal Borel subsuperalgebra* of \mathfrak{g} .

Proposition 3.4. *If \mathfrak{g} is isomorphic to either $H(n)$, $S(n)$, or $W(n)$, then \mathfrak{g} admits at least one triangular decomposition satisfying (C2).*

Proof. We claim that there exists a triangular decomposition of \mathfrak{g} for which the highest root of \mathfrak{g} is a root of \mathfrak{g}_0 . Indeed, since \mathfrak{g} is simple we have that $\mathfrak{g} \cong L_{\mathfrak{b}_{\min}}(\lambda)$ as a \mathfrak{g} -module, where λ is its highest root with respect to \mathfrak{b}_{\min} . We now prove our claim case by case:

Let $\mathfrak{g} = W(n)$. Then we have that $\lambda = -\varepsilon_n$ and the unique odd simple root for \mathfrak{b}_{\min} is $\alpha = -\varepsilon_1$. Set $\mathfrak{b}_1 = r_\alpha(\mathfrak{b}_{\min})$. Since

$$\mathfrak{h}_{-\varepsilon_i} = \{h \in \mathfrak{h} \mid \varepsilon_i(h) = 0\}, \quad (\text{precisely } \mathfrak{h}_{-\varepsilon_i} = \text{span}_{\mathbb{K}}\{\xi_j \partial_j \mid j = 1, \dots, n, j \neq i\})$$

we have that $\varepsilon_n(\mathfrak{h}_{-\varepsilon_1}) \neq 0$ and hence $L_{\mathfrak{b}_{\min}}(-\varepsilon_n) \cong L_{\mathfrak{b}_1}(-\varepsilon_n + \varepsilon_1)$ (see [Ser05, Lemma 5.3]). To conclude this case, we notice that $-\varepsilon_n$ is a root of \mathfrak{g}_{-1} and ε_1 is a root of \mathfrak{g}_1 , which implies that $(-\varepsilon_n + \varepsilon_1)$ is a root of \mathfrak{g}_0 , as we want.

The proof for $W(n)$ also works for $S(n)$. The only difference is that

$$\mathfrak{h}_{-\varepsilon_i} = \{h \in \mathfrak{h} \mid \varepsilon_i(h) = 0, (\varepsilon_1 + \dots + \varepsilon_n)(h) = 0\},$$

but it is still clear that $\varepsilon_n(\mathfrak{h}_{-\varepsilon_1}) \neq 0$ and hence $L_{\mathfrak{b}_{\min}}(-\varepsilon_n) \cong L_{\mathfrak{b}_1}(-\varepsilon_n + \varepsilon_1)$.

Finally, suppose that $\mathfrak{g} = H(n)$. For $n = 2k$, we have that $\lambda = \varepsilon_1 - \delta$ and the unique odd simple root for \mathfrak{b}_{\min} is $\alpha_1 = -\varepsilon_1 - \delta$. Set $\mathfrak{b}_1 = r_{\alpha_1}(\mathfrak{b}_{\min})$. Since

$$\mathfrak{h}_{\varepsilon_i - \delta} = \mathfrak{h}_{-\varepsilon_i - \delta} = \{h \in \mathfrak{h} \mid \varepsilon_i(h) = \delta(h) = 0\},$$

we have that $\lambda(\mathfrak{h}_{-\varepsilon_1 - \delta}) = 0$. Hence $L_{\mathfrak{b}_{\min}}(\lambda) \cong L_{\mathfrak{b}_1}(\lambda)$. Now, $\alpha_2 = -\varepsilon_2 - \delta$ is an odd simple root for \mathfrak{b}_1 . Since $\lambda(\mathfrak{h}_{-\varepsilon_1 - \delta}) \neq 0$, we obtain that $L_{\mathfrak{b}_{\min}}(\lambda) \cong L_{\mathfrak{b}_2}(\lambda - \alpha_2)$, where $\mathfrak{b}_2 = r_{\alpha_2}(\mathfrak{b}_1)$. In particular, the highest root of \mathfrak{g} with respect to \mathfrak{b}_2 is $\varepsilon_1 + \varepsilon_2$, which is clearly a root of \mathfrak{g}_0 . Now, we suppose that $n = 2k + 1$. Observing that $\mathfrak{h}_{\pm\varepsilon_i - \delta}$ are the same as in the case $n = 2k$, we have that the proof of the case $n = 2k$ also works for $n = 2k + 1$. This proves the claim.

We conclude by noting that for all cases we have found a triangular decomposition for which the highest root of \mathfrak{g} is in R_0 . Therefore the opposite triangular decomposition of the one we found satisfies the desired condition. Now, to conclude the proof we notice that the set of negative roots with respect to the found triangular decompositions is as follows:

$$\begin{aligned} R^- &= R_0^- \cup (R_{-1} \setminus \{-\varepsilon_n\}) \cup \{\varepsilon_n\}, & \text{for } \mathfrak{g} = W(n) \text{ or } S(n), \\ R^- &= R_0^- \cup (R_{-1} \setminus \{-\varepsilon_1 - \delta, -\varepsilon_2 - \delta\}) \cup \{\varepsilon_1 + \delta, \varepsilon_2 + \delta\}, & \text{for } \mathfrak{g} = H(n). \end{aligned}$$

In particular, $\mathfrak{n}_0^- \subseteq \mathfrak{r}$ and hence such triangular decompositions satisfy (C1). \square

3.3. Periplectic Lie superalgebras. For each $n \geq 2$, let $\mathfrak{p}(n)$ be the Lie subsuperalgebra of $\mathfrak{gl}(n+1, n+1)$ consisting of all matrices of the form

$$(3.4) \quad M = \left(\begin{array}{c|c} A & B \\ \hline C & -A^t \end{array} \right), \quad \text{where } A \in \mathfrak{sl}(n+1), B = B^t \text{ and } C^t = -C.$$

Throughout this subsection, \mathfrak{g} will denote $\mathfrak{p}(n)$. Notice that \mathfrak{g}_0 is isomorphic to the Lie algebra $\mathfrak{sl}(n+1)$, and as a \mathfrak{g}_0 -module, the structure of \mathfrak{g}_1 is the following. Let $S^2(\mathbb{K}^{n+1})$ (resp. $\Lambda^2(\mathbb{K}^{n+1})^*$) denote the second symmetric (resp. exterior) power of \mathbb{K}^{n+1} (resp. $(\mathbb{K}^{n+1})^*$), with the natural action of $\mathfrak{sl}(n+1)$ (by matrix multiplication) in each term. Denote by \mathfrak{g}_1^+ (resp. \mathfrak{g}_1^-) the set of all matrices of the form (3.4) such that $A = C = 0$ (resp. $A = B = 0$), and notice that, as \mathfrak{g}_0 -modules, $\mathfrak{g}_1 \cong \mathfrak{g}_1^+ \oplus \mathfrak{g}_1^-$, where $\mathfrak{g}_1^+ \cong S^2(\mathbb{K}^{n+1})$ and $\mathfrak{g}_1^- \cong \Lambda^2(\mathbb{K}^{n+1})^*$.

Consider $\mathfrak{g}_{-1} = \mathfrak{g}_1^-$, $\mathfrak{g}_0 = \mathfrak{g}_0$ and $\mathfrak{g}_1 = \mathfrak{g}_1^+$. Then $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a \mathbb{Z} -grading of \mathfrak{g} that is compatible with the \mathbb{Z}_2 -grading ($\mathfrak{g}_0 = \mathfrak{g}_0$ and $\mathfrak{g}_1 = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$). Let $\mathfrak{h} \subseteq \mathfrak{g}_0$ be a Cartan subalgebra of \mathfrak{g}_0 , recall that \mathfrak{g}_0 is isomorphic to \mathfrak{sl}_{n+1} , and identify \mathfrak{h} with \mathfrak{h}^* via the Killing form $(A_1, A_2) = \text{tr}(A_1 A_2)$. If $\{\varepsilon_1, \dots, \varepsilon_n\}$ is the standard orthogonal basis of \mathfrak{h} , then the roots of \mathfrak{g} are described as follows:

- Roots of \mathfrak{g}_{-1} : $-\varepsilon_i - \varepsilon_j$, where $1 \leq i < j \leq n$.
- Roots of \mathfrak{g}_0 : $\varepsilon_i - \varepsilon_j$, where $i \neq j$ and $1 \leq i, j \leq n$.
- Roots of \mathfrak{g}_1 : $\varepsilon_i + \varepsilon_j$, where $1 \leq i \leq j \leq n$.

Consider the triangular decomposition

$$\mathfrak{n}_0^- \oplus \mathfrak{h} \oplus \mathfrak{n}_0^+, \quad \text{where } \mathfrak{n}_0^\pm = \bigoplus_{1 \leq i < j \leq n} \mathfrak{g}_{\pm(\varepsilon_i - \varepsilon_j)}.$$

This triangular decomposition induces a triangular decomposition on \mathfrak{g} and we have the following result.

Proposition 3.5. *If \mathfrak{g} is isomorphic to $\mathfrak{p}(n)$ with $n \geq 2$, then the triangular decomposition*

$$\mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+, \quad \text{where } \mathfrak{n}^\pm = \mathfrak{g}_{\pm 1} \oplus \mathfrak{n}_0^\pm$$

satisfies (C1). In particular, $\mathfrak{n}_0^- = \mathfrak{n}_0^-$ and all the roots of \mathfrak{g}_1 are positive.

As usual, let \mathfrak{b} be the Borel subsuperalgebra $\mathfrak{h} \oplus \mathfrak{n}^+ \subseteq \mathfrak{g}$, R be the set of roots of \mathfrak{g} , R^+ be the positive roots associated to this decomposition, etc. Notice that $R^- \neq -R^+$, since, for each $i \in \{1, \dots, n\}$, there exists a positive root of the form $2\varepsilon_i$, such that $-\alpha \notin R$.

3.4. Remarks on triangular decompositions. Recall conditions (C1) and (C2).

Theorem 3.6. *If \mathfrak{g} is either $\mathfrak{sl}(n, n)$ with $n \geq 2$, or a finite-dimensional simple Lie superalgebra not of type $\tilde{S}(n)$ or $\mathfrak{q}(n)$, then \mathfrak{g} admits at least one triangular decomposition satisfying (C1). Moreover, if $\mathfrak{g} \not\cong \mathfrak{p}(n)$, then \mathfrak{g} also admits at least one triangular decomposition satisfying (C2).*

Proof. This result follows from Propositions 3.1, 3.4 and 3.5. □

Remark 3.7. (a) If \mathfrak{g} is basic classical, then every triangular decomposition satisfies (C1), since $\mathfrak{r} = \mathfrak{g}_0$. Moreover, if \mathfrak{g} is of type II, then every distinguished triangular decomposition also satisfies (C2). However, if \mathfrak{g} is of type I, then its distinguished triangular decomposition satisfies (C1) but not (C2), as the lowest root of \mathfrak{g} is a root of \mathfrak{g}_{-1} .

- (b) If \mathfrak{g} is of Cartan type, then there are triangular decompositions that do not satisfy (C1). For instance, the triangular decomposition induced by \mathfrak{b}_{\min} , as $\mathfrak{r} \subsetneq \mathfrak{g}_{\bar{0}}$. Moreover, there are triangular decompositions satisfying (C1) but not (C2). For instance, the triangular decomposition induced by \mathfrak{b}_{\max} , since the lowest root of \mathfrak{g} with respect this triangular decomposition is a root of $\mathfrak{g}_{-1} \subseteq \mathfrak{g}_{\bar{1}}$.
- (c) If $\mathfrak{g} = \mathfrak{p}(n)$, then every triangular decomposition satisfies (C1). However, in this case, we could not find a method to construct triangular decompositions satisfying (C2). The difficulty in this case is the nonexistence of odd reflections.
- (d) For the remainder of this paper we will always consider triangular decompositions satisfying (C1) for Lie superalgebras of types $\mathfrak{p}(n)$, $H(n)$, $S(n)$ and $W(n)$, unless otherwise specified.

4. GENERALIZED KAC MODULES

In this section, assume that \mathfrak{g} is either $\mathfrak{sl}(n, n)$ with $n \geq 2$, or a finite-dimensional simple Lie superalgebra not of type $\tilde{S}(n)$ or $\mathfrak{q}(n)$. Let \mathfrak{h} be a fixed Cartan subalgebra of \mathfrak{g} (that is, a Cartan subalgebra of \mathfrak{r}), R be the set of roots of \mathfrak{g} (with respect to \mathfrak{h}), and $Q \subseteq \mathfrak{h}^*$ the root lattice $\sum_{\alpha \in R} \mathbb{Z}\alpha$. Fix a set of simple roots $\Delta \subseteq R$, let $R^+ \subseteq R$ be the set of positive roots, Q^+ be the positive cone $\sum_{\alpha \in R^+} \mathbb{N}\alpha$ and $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be the associated triangular decomposition of \mathfrak{g} . Notice that $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ is a solvable subsuperalgebra of \mathfrak{g} , that \mathfrak{n}^\pm are nilpotent ideals of \mathfrak{b} , and that \mathfrak{h}^* admits a partial order given by: $\lambda \leq \mu \in \mathfrak{h}^*$ if and only if $\mu - \lambda \in Q^+$.

Let $R_{\mathfrak{r}}$ denote the root system $\{\alpha \in \mathfrak{h}^* \mid \tau_\alpha \neq \{0\}, \alpha \neq 0\}$ of \mathfrak{r} , $R_{\mathfrak{r}}^+$ be the positive system $R^+ \cap R_{\mathfrak{r}}$, and $\Delta_{\mathfrak{r}}$ be the simple system associated to $R_{\mathfrak{r}}^+$. Since \mathfrak{r}' is a finite-dimensional semisimple Lie algebra, for each $\alpha \in R_{\mathfrak{r}}^+$, one can choose elements $x_\alpha \in \mathfrak{r}_\alpha$, $x_\alpha^- \in \mathfrak{r}_{-\alpha}$, and $h_\alpha \in \mathfrak{h}$, such that: the subalgebra generated by $\{x_\alpha^-, h_\alpha, x_\alpha\}$ is isomorphic to $\mathfrak{sl}(2)$, $[x_\alpha, x_\alpha^-] = h_\alpha$, $[h_\alpha, x_\alpha^-] = -2x_\alpha$, and $[h_\alpha, x_\alpha] = 2x_\alpha$. The triple $(x_\alpha, x_\alpha^-, h_\alpha)$ is said to be an $\mathfrak{sl}(2)$ -triple and the Lie subalgebra generated by $\{x_\alpha^-, h_\alpha, x_\alpha\}$ will be denoted \mathfrak{sl}_α . Recall that $X^+ = \{\lambda \in \mathfrak{h}^* \mid L_{\mathfrak{b}}(\lambda) \text{ is finite dimensional}\}$, and notice that, for $\lambda \in X^+$, we have $\lambda(h_\alpha) \in \mathbb{N}$, for all $\alpha \in \Delta_{\mathfrak{r}}$ (since $L_{\mathfrak{b}}(\lambda)$ is also a finite-dimensional \mathfrak{r}' -module).

Definition 4.1 (Generalized Kac module). Let $\lambda \in X^+$. The *generalized Kac module* associated to λ is defined to be the cyclic \mathfrak{g} -module $K(\lambda)$ given as a quotient of $\mathbf{U}(\mathfrak{g})$ by the left ideal generated by

$$\mathfrak{n}^+, \quad h - \lambda(h), \quad (x_\alpha^-)^{\lambda(h_\alpha)+1}, \quad \text{for all } h \in \mathfrak{h} \text{ and } \alpha \in \Delta_{\mathfrak{r}}.$$

Denote the image of $1 \in \mathbf{U}(\mathfrak{g})$ in $K(\lambda)$ by k_λ , and notice that as a \mathfrak{g} -module, $K(\lambda)$ is generated by the vector $k_\lambda \in K(\lambda)_{\bar{0}}$, satisfying the following defining relations

$$(4.1) \quad \mathfrak{n}^+ k_\lambda = 0, \quad h k_\lambda = \lambda(h) k_\lambda, \quad (x_\alpha^-)^{\lambda(h_\alpha)+1} k_\lambda = 0, \quad \text{for all } h \in \mathfrak{h} \text{ and } \alpha \in \Delta_{\mathfrak{r}}.$$

If \mathfrak{g} is a basic classical Lie superalgebra or $\mathfrak{g} \cong \mathfrak{sl}(n, n)$, $n \geq 2$, then $K(\lambda) = \bar{V}(\lambda)$ [CLS, Definition 2.6]. The results in this section generalize those proved for basic classical Lie superalgebras in [CLS, §2.4].

Proposition 4.2. *For all $\lambda \in X^+$, the \mathfrak{g} -module $K(\lambda)$ is finite dimensional.*

Proof. If \mathfrak{g} is either a basic classical Lie superalgebra or $\mathfrak{g} \cong \mathfrak{sl}(n, n)$ with $n \geq 2$, the result was proved in [CLS, Proposition 2.7]. So we assume that \mathfrak{g} is a finite-dimensional Lie superalgebra either of Cartan type, or isomorphic to $\mathfrak{p}(n)$.

Since $\lambda(h_\alpha) \in \mathbb{N}$ for all $\alpha \in \Delta_{\mathfrak{r}}$, we can consider the finite-dimensional irreducible \mathfrak{r} -module of highest weight λ , $L_0(\lambda)$. Recall that \mathfrak{r} is a reductive Lie algebra and \mathfrak{z} acts as a scalar on $L_0(\lambda)$. Hence $L_0(\lambda)$ is isomorphic to the \mathfrak{r} -module generated by a vector u_λ with defining relations

$$x_\alpha u_\lambda = 0, \quad hu_\lambda = \lambda(h)u_\lambda, \quad (x_\alpha^-)^{\lambda(h_\alpha)+1}u_\lambda = 0, \quad \text{for all } h \in \mathfrak{h} \text{ and } \alpha \in \Delta_{\mathfrak{r}}.$$

Let $W = \mathbf{U}(\mathfrak{r})k_\lambda$ be the \mathfrak{r} -submodule of $K(\lambda)$ generated by k_λ . Since W is cyclic and k_λ satisfies (4.1), there exists a unique (surjective) homomorphism of \mathfrak{r} -modules satisfying

$$\varphi: L_0(\lambda) \twoheadrightarrow W, \quad u_\lambda \mapsto k_\lambda.$$

Since φ is surjective and $L_0(\lambda)$ is finite dimensional, this shows that W is finite dimensional.

Now, recall that $\mathfrak{n}_0^- = \mathfrak{n}_0^- \subseteq \mathfrak{r}$ (see Remark 3.7(d)). Thus, by the PBW Theorem,

$$K(\lambda) = \mathbf{U}(\mathfrak{g})k_\lambda = \mathbf{U}(\mathfrak{n}_1^-)\mathbf{U}(\mathfrak{n}_0^-)k_\lambda = \mathbf{U}(\mathfrak{n}_1^-)W,$$

where $\mathfrak{n}_1^- = \mathfrak{n}^- \cap \mathfrak{g}_1^-$. Moreover, for any given basis $\{x_i \mid 1 \leq i \leq \dim \mathfrak{n}_1^-\} \subseteq \mathfrak{n}_1^-$, we have that the set

$$\{x_{j_1} \cdots x_{j_k} \mid 1 \leq j_1 < \cdots < j_k \leq \dim \mathfrak{n}_1^-\}$$

forms a (finite) basis for $\mathbf{U}(\mathfrak{n}_1^-)$. Thus, we conclude that $K(\lambda)$ is finite dimensional. \square

Lemma 4.3. *Let $\lambda \in X^+$. If V is a finite-dimensional \mathfrak{g} -module generated by a highest-weight vector of weight λ , then there exists a surjective homomorphism of \mathfrak{g} -modules $\pi_V: K(\lambda) \twoheadrightarrow V$. Moreover, there exists a unique \mathfrak{g} -submodule $W \subseteq K(\lambda)$ such that $V \cong K(\lambda)/W$.*

Proof. This proof is similar to the proof of [CLS, Lemma 2.8]. \square

Since every irreducible finite-dimensional \mathfrak{g} -module is generated by a highest-weight vector of weight $\lambda \in X^+$, Lemma 4.3 applies, in particular, to all irreducible finite-dimensional \mathfrak{g} -modules.

5. GLOBAL WEYL MODULES

Let \mathfrak{g} be a Lie superalgebra and consider an associative commutative \mathbb{k} -algebra A with unit. The vector space $\mathfrak{g} \otimes_{\mathbb{k}} A$ is a Lie superalgebra when endowed with the \mathbb{Z}_2 -grading given by $(\mathfrak{g} \otimes A)_j = \mathfrak{g}_j \otimes A$, $j \in \mathbb{Z}_2$, and the Lie superbracket extending bilinearly

$$[x_1 \otimes a_1, x_2 \otimes a_2] = [x_1, x_2] \otimes a_1 a_2, \quad \text{for all } x_1, x_2 \in \mathfrak{g} \text{ and } a_1, a_2 \in A.$$

We refer to a Lie superalgebra of this form as a *map Lie superalgebra*. From now on, we identify \mathfrak{g} with a subsuperalgebra of $\mathfrak{g} \otimes A$ via the isomorphism $\mathfrak{g} \cong \mathfrak{g} \otimes \mathbb{k}$ and the inclusion $\mathfrak{g} \otimes \mathbb{k} \subseteq \mathfrak{g} \otimes A$.

Let \mathfrak{g} be either $\mathfrak{sl}(n, n)$ with $n \geq 2$, or a finite-dimensional simple Lie superalgebra not of type $\tilde{S}(n)$ or $\mathfrak{q}(n)$. Let \mathcal{C}_A denote the category of $\mathfrak{g} \otimes A$ -modules that are finitely semisimple as \mathfrak{r} -modules (see (2.1) for the notation). By Lemma 2.5, $\mathcal{C}_A = \mathcal{C}_{(\mathfrak{g} \otimes A, \mathfrak{r})}$ is an abelian category, closed under taking submodules, quotients, arbitrary direct sums, and finite tensor products.

Lemma 5.1. *If \mathfrak{g} is either $\mathfrak{sl}(n, n)$ with $n \geq 2$, or a finite-dimensional simple Lie superalgebra not of type $\tilde{S}(n)$ or $\mathfrak{q}(n)$, then \mathfrak{g} is a finite-dimensional completely reducible \mathfrak{r} -module via the adjoint representation.*

Proof. First notice that \mathfrak{g} is finite dimensional by hypothesis. Also notice that \mathfrak{r} is a completely reducible \mathfrak{r} -module via the adjoint representation, since it is a finite-dimensional reductive Lie algebra. If \mathfrak{g} is a basic classical Lie superalgebra, or $\mathfrak{sl}(n, n)$ with $n \geq 2$, then \mathfrak{g}_1 is a completely reducible \mathfrak{r} -module. Since \mathfrak{r} is also a completely reducible \mathfrak{r} -module and $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{g}_1$, we have that \mathfrak{g} is a completely reducible \mathfrak{r} -module. If \mathfrak{g} is $\mathfrak{p}(n)$, $S(n)$, or $H(n)$, then \mathfrak{r} is a finite-dimensional simple

Lie algebra and, since \mathfrak{g} is finite dimensional, it is a completely reducible \mathfrak{r} -module. If $\mathfrak{g} = W(n)$, recall that $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \cdots \oplus \mathfrak{g}_{n-1}$, with $\mathfrak{g}_0 = \mathfrak{r} = \mathfrak{z} \oplus \mathfrak{r}'$, where \mathfrak{z} is the one-dimensional center and \mathfrak{r}' is the semisimple part of \mathfrak{r} . Furthermore, for each $i \in \{1, \dots, n-1\}$, \mathfrak{g}_i is a semisimple \mathfrak{r}' -module and \mathfrak{z} acts on \mathfrak{g}_i by a scalar (see [Kac77, Proposition 1.2.12]). Hence, for each $i \in \{1, \dots, n-1\}$, the decomposition of \mathfrak{g}_i into a direct sum of irreducible \mathfrak{r}' -modules is also a decomposition into a direct sum of irreducible \mathfrak{r} -modules. Thus, \mathfrak{g} is a completely reducible \mathfrak{r} -module. \square

Lemma 5.2. *If V is a finitely-semisimple \mathfrak{r} -module, then $\text{ind}_{\mathfrak{r}}^{\mathfrak{g} \otimes A} V$ is a projective object in \mathcal{C}_A . Moreover, the category \mathcal{C}_A has enough projectives.*

Proof. This proof follows from Lemma 5.1 and results of Section 2 using standard arguments. \square

Given a \mathfrak{g} -module V , define $P_A(V)$ to be the $\mathfrak{g} \otimes A$ -module

$$(5.1) \quad P_A(V) = \text{ind}_{\mathfrak{g}}^{\mathfrak{g} \otimes A} V.$$

By Corollary 2.3, if V is a projective \mathfrak{g} -module, then $P_A(V)$ is a projective $\mathfrak{g} \otimes A$ -module.

The next result, which was proved in [CLS, Proposition 3.2] for the cases where \mathfrak{g} is either basic classical or $\mathfrak{sl}(n, n)$ with $n \geq 2$, describes $P_A(K(\lambda))$ by generators and relations.

Proposition 5.3. *If $\lambda \in X^+$, then $P_A(K(\lambda))$ is generated as a left $\mathbf{U}(\mathfrak{g} \otimes A)$ -module, by a vector $p_\lambda \in P_A(K(\lambda))_{\bar{0}}$ satisfying the following defining relations*

$$(5.2) \quad \mathfrak{n}^+ p_\lambda = 0, \quad h p_\lambda = \lambda(h) p_\lambda, \quad (x_\alpha^-)^{\lambda(h_\alpha)+1} p_\lambda = 0, \quad \text{for all } h \in \mathfrak{h} \text{ and } \alpha \in \Delta_{\mathfrak{r}}.$$

Proof. Let $p_\lambda = 1 \otimes k_\lambda \in P_A(K(\lambda))$. Since $k_\lambda \in K(\lambda)_{\bar{0}}$ satisfies relations (4.1), $p_\lambda \in P_A(K(\lambda))_{\bar{0}}$ satisfies relations (5.2). The fact that these are defining relations follows from Lemma 2.7. \square

Given $\lambda \in X^+$ and $M \in \mathcal{C}_A$, define

$$(5.3) \quad M^\lambda = M / \sum_{\mu \leq \lambda} \mathbf{U}(\mathfrak{g} \otimes A) M_\mu.$$

Notice that, if μ is a weight of M^λ , then $\mu \leq \lambda$. Let \mathcal{C}_A^λ denote the full subcategory of \mathcal{C}_A whose objects are the left $\mathbf{U}(\mathfrak{g} \otimes A)$ -modules $M \in \mathcal{C}_A$ such that $M^\lambda = M$.

Lemma 5.4. *Let $\lambda \in X^+$ and V be a \mathfrak{g} -module. If V is finitely semisimple as an \mathfrak{r} -module, then $P_A(V)^\lambda$ is a projective object in \mathcal{C}_A^λ . Moreover, the category \mathcal{C}_A^λ has enough projectives.*

Proof. First recall from the construction of \mathcal{C}_A^λ , that

$$\text{Hom}_{\mathcal{C}_A^\lambda}(P_A(V)^\lambda, M) = \text{Hom}_{\mathcal{C}_A}(P_A(V)^\lambda, M) \quad \text{for every object } M \text{ of } \mathcal{C}_A^\lambda.$$

Now, since morphisms in \mathcal{C}_A are homomorphisms of \mathfrak{h} -modules and $\mathfrak{g} \otimes A$ -modules, precomposition with the quotient (5.3) gives an isomorphism of vector spaces

$$\text{Hom}_{\mathcal{C}_A}(P_A(V)^\lambda, M) \cong \text{Hom}_{\mathcal{C}_A}(P_A(V), M) \quad \text{for every object } M \text{ of } \mathcal{C}_A^\lambda,$$

which is functorial in M . Finally, recall from Lemma 5.2 that $P_A(V)$ is a projective object of \mathcal{C}_A . This shows that $P_A(V)^\lambda$ is a projective object in \mathcal{C}_A^λ . The proof that \mathcal{C}_A^λ has enough projectives is similar to the proof of Lemma 5.2 and thus will be omitted. \square

Definition 5.5 (Global Weyl module). Let $\lambda \in X^+$. The *global Weyl module* associated to λ is defined to be

$$W_A(\lambda) := P_A(K(\lambda))^\lambda.$$

The image of p_λ in $W_A(\lambda)$ will be denoted by w_λ .

The next result, which was proved in [CFK10, Proposition 4] for the non-super setting and in [CLS, Proposition 3.4] for the case where \mathfrak{g} is either basic classical or $\mathfrak{sl}(n, n)$ with $n \geq 2$, provides a description of global Weyl modules by generators and relations.

Proposition 5.6. *For $\lambda \in X^+$, the global Weyl module $W_A(\lambda)$ is generated as a left $\mathbf{U}(\mathfrak{g} \otimes A)$ -module, by the vector w_λ , with defining relations*

$$(5.4) \quad (\mathfrak{n}^+ \otimes A)w_\lambda = 0, \quad hw_\lambda = \lambda(h)w_\lambda, \quad (x_\alpha^-)^{\lambda(h_\alpha)+1}w_\lambda = 0, \quad \text{for all } h \in \mathfrak{h} \text{ and } \alpha \in \Delta_{\mathfrak{r}}.$$

Proof. Recall that, if μ is a weight of $W_A(\lambda)$, then $\mu \leq \lambda$. Thus $(\mathfrak{n}^+ \otimes A)w_\lambda = 0$. The remaining relations are satisfied by w_λ , since they are satisfied by p_λ (see Proposition 5.3). The proof that these are the only relations is similar to that in the proof of [CLS, Proposition 3.4]. \square

Proposition 5.7. *Let $\lambda \in X^+$. The global Weyl module $W_A(\lambda)$ is the unique object of \mathcal{C}_A^λ , up to isomorphism, that is generated by a highest-weight vector of weight λ and admits a surjective homomorphism onto every object of \mathcal{C}_A^λ that is generated by a highest-weight vector of weight λ .*

Proof. This result was proved in [CLS, Proposition 3.5] for the case where \mathfrak{g} is either basic classical or $\mathfrak{sl}(n, n)$ with $n \geq 2$. The proof for the other types is similar. \square

When $A = \mathbb{k}$, the global Weyl module $W_A(\lambda)$ coincides with the generalized Kac module $K(\lambda)$. In this case, Proposition 5.7 reduces to the universal property given in Lemma 4.3.

6. WEYL FUNCTORS

Let \mathfrak{g} be either $\mathfrak{sl}(n, n)$ with $n \geq 2$, or a finite-dimensional simple Lie superalgebra not of type $\tilde{S}(n)$ or $\mathfrak{q}(n)$, and let A be an associative commutative \mathbb{k} -algebra with unit. In this section we will define *Weyl functors* for Lie superalgebras. These generalize the Weyl functors defined in [CFK10, p. 525].

Let $\lambda \in X^+$. Recall from Definition 5.5, that w_λ denotes the image of p_λ in $W_A(\lambda)$, and set

$$\text{Ann}_{\mathfrak{g} \otimes A}(w_\lambda) = \{u \in \mathbf{U}(\mathfrak{g} \otimes A) \mid uw_\lambda = 0\},$$

$$\text{Ann}_{\mathfrak{h} \otimes A}(w_\lambda) = \text{Ann}_{\mathfrak{g} \otimes A}(w_\lambda) \cap \mathbf{U}(\mathfrak{h} \otimes A).$$

Notice that $\text{Ann}_{\mathfrak{g} \otimes A}(w_\lambda)$ is a left ideal of $\mathbf{U}(\mathfrak{g} \otimes A)$, and thus, since $\mathbf{U}(\mathfrak{h} \otimes A)$ is a commutative algebra, $\text{Ann}_{\mathfrak{h} \otimes A}(w_\lambda)$ is an ideal of $\mathbf{U}(\mathfrak{h} \otimes A)$. Define the algebra \mathbf{A}_λ to be the quotient

$$\mathbf{A}_\lambda = \mathbf{U}(\mathfrak{h} \otimes A) / \text{Ann}_{\mathfrak{h} \otimes A}(w_\lambda).$$

Remark 6.1. By Proposition 5.7, $W_A(\lambda)_\lambda$ is isomorphic to $\mathbf{U}(\mathfrak{g} \otimes A) / \text{Ann}_{\mathfrak{g} \otimes A}(w_\lambda)$ as a $\mathfrak{g} \otimes A$ -module. Moreover, by the PBW Theorem, $W_A(\lambda) = \mathbf{U}(\mathfrak{h} \otimes A)w_\lambda$. Thus, the unique homomorphism of $\mathbf{U}(\mathfrak{h} \otimes A)$ -modules satisfying

$$\phi: \mathbf{U}(\mathfrak{h} \otimes A) \rightarrow W_A(\lambda)_\lambda, \quad \phi(1) = w_\lambda$$

induces an isomorphism of $\mathfrak{h} \otimes A$ -modules between $W_A(\lambda)_\lambda$ and $\mathbf{U}(\mathfrak{h} \otimes A) / \text{Ann}_{\mathfrak{h} \otimes A}(w_\lambda)$. In other words, $W_A(\lambda)_\lambda \cong \mathbf{A}_\lambda$ as right \mathbf{A}_λ -modules.

Lemma 6.2. *For all $\lambda \in X^+$ and $V \in \mathcal{C}_A^\lambda$, we have $(\text{Ann}_{\mathfrak{h} \otimes A}(w_\lambda))V_\lambda = 0$.*

Proof. Let $v \in V_\lambda$ and $W = \mathbf{U}(\mathfrak{g} \otimes A)v$. Since V is an object of \mathcal{C}_A^λ , the submodule W is also an object of \mathcal{C}_A^λ (Lemma 2.5). Moreover, since $v \in V_\lambda$, we have $(\mathfrak{n}^+ \otimes A)v = 0$ and $hv = \lambda(h)v$ for all $h \in \mathfrak{h}$. Thus, by Proposition 5.7, there exists a unique (surjective) homomorphism of $\mathfrak{g} \otimes A$ -modules $\pi: W_A(\lambda) \twoheadrightarrow W$ satisfying $\pi(w_\lambda) = v$. Since π is a homomorphism of $\mathfrak{g} \otimes A$ -modules and $uw_\lambda = 0$ for all $u \in \text{Ann}_{\mathfrak{h} \otimes A}(w_\lambda)$, we conclude that $uv = \pi(uw_\lambda) = 0$ for all $u \in \text{Ann}_{\mathfrak{h} \otimes A}(w_\lambda)$. \square

Recall that $\mathbf{U}(\mathfrak{h} \otimes A)$ is a commutative algebra, so every left $\mathbf{U}(\mathfrak{h} \otimes A)$ -module is naturally a right $\mathbf{U}(\mathfrak{h} \otimes A)$ -module. Given $\lambda \in X^+$, Lemma 6.2 implies that the left action of $\mathbf{U}(\mathfrak{g} \otimes A)$ on any object V of \mathcal{C}_A^λ induces a left (as well as a right) action of \mathbf{A}_λ on V_λ . Since $W_A(\lambda)$ is an object of \mathcal{C}_A^λ generated by $w_\lambda \in W_A(\lambda)_\lambda$ as a left $\mathbf{U}(\mathfrak{g} \otimes A)$ -module, we have a right action of \mathbf{A}_λ on $W_A(\lambda)$ that commutes with the left $\mathbf{U}(\mathfrak{g} \otimes A)$ action; namely,

$$(6.1) \quad (uw_\lambda)u' = uu'w_\lambda \quad \text{for all } u \in \mathbf{U}(\mathfrak{g} \otimes A) \text{ and } u' \in \mathbf{U}(\mathfrak{h} \otimes A).$$

Thus, with these actions, $W_A(\lambda)$ is a $(\mathbf{U}(\mathfrak{g} \otimes A), \mathbf{A}_\lambda)$ -bimodule.

Given $\lambda \in X^+$, let $\mathbf{A}_\lambda\text{-mod}$ denote the category of left \mathbf{A}_λ -modules and let $M \in \mathbf{A}_\lambda\text{-mod}$. Since $W_A(\lambda)$ is a finitely-semisimple \mathfrak{r} -module and the action of \mathfrak{r} on $W_A(\lambda) \otimes_{\mathbf{A}_\lambda} M$ is given by left multiplication, we have that $W_A(\lambda) \otimes_{\mathbf{A}_\lambda} M$ is a finitely semisimple \mathfrak{r} -module. Since $\text{id}: W_A(\lambda) \rightarrow W_A(\lambda)$ is an even homomorphism of $\mathfrak{g} \otimes A$ -modules, for every $M, M' \in \mathbf{A}_\lambda\text{-mod}$ and $f \in \text{Hom}_{\mathbf{A}_\lambda}(M, M')$,

$$\text{id} \otimes f: W_A(\lambda) \otimes_{\mathbf{A}_\lambda} M \rightarrow W_A(\lambda) \otimes_{\mathbf{A}_\lambda} M'$$

is a homomorphism of $\mathfrak{g} \otimes A$ -modules.

Definition 6.3 (Weyl functor). Let $\lambda \in X^+$. The Weyl functor associated to λ is defined to be

$$\mathbf{W}_A^\lambda: \mathbf{A}_\lambda\text{-mod} \rightarrow \mathcal{C}_A^\lambda, \quad \mathbf{W}_A^\lambda M = W_A(\lambda) \otimes_{\mathbf{A}_\lambda} M, \quad \mathbf{W}_A^\lambda f = \text{id}_{W_A(\lambda)} \otimes f,$$

for all M, M' in $\mathbf{A}_\lambda\text{-mod}$ and $f \in \text{Hom}_{\mathbf{A}_\lambda}(M, M')$.

Given $\lambda \in X^+$, notice that there is an isomorphism of $\mathfrak{g} \otimes A$ -modules $\mathbf{W}_A^\lambda \mathbf{A}_\lambda \cong W_A(\lambda)$. Also notice that, for all $\mu \in \mathfrak{h}^*$ and M in $\mathbf{A}_\lambda\text{-mod}$, we have

$$(6.2) \quad (\mathbf{W}_A^\lambda M)_\mu = W_A(\lambda)_\mu \otimes_{\mathbf{A}_\lambda} M.$$

Given $\lambda \in X^+$, recall that Lemma 6.2 implies that $W_A(\lambda)$ is a $(\mathbf{U}(\mathfrak{g} \otimes A), \mathbf{A}_\lambda)$ -bimodule. This implies in particular, that $\text{Hom}_{\mathcal{C}_A^\lambda}(W_A(\lambda), N)$ can be viewed as an \mathbf{A}_λ -module for any object N of \mathcal{C}_A^λ via

$$(u \cdot f)(v) = f(v \cdot u) \quad \text{for all } u \in \mathbf{A}_\lambda, f \in \text{Hom}_{\mathcal{C}_A^\lambda}(W_A(\lambda), N) \text{ and } v \in W_A(\lambda).$$

Moreover, Lemma 6.2 also implies that the left action of $\mathbf{U}(\mathfrak{g} \otimes A)$ on an object V in \mathcal{C}_A^λ induces a left action of \mathbf{A}_λ on V_λ .

Lemma 6.4. *Let $\lambda \in X^+$. For every object N of \mathcal{C}_A^λ , the map*

$$\text{Hom}_{\mathcal{C}_A^\lambda}(W_A(\lambda), N) \rightarrow N_\lambda, \quad f \mapsto f(w_\lambda)$$

is an isomorphism of \mathbf{A}_λ -modules that is functorial in N .

Proof. Fix an object N in \mathcal{C}_A^λ and a homomorphism $f \in \text{Hom}_{\mathcal{C}_A^\lambda}(W_A(\lambda), N)$. First notice that, since $w_\lambda \in W_A(\lambda)_\lambda$, then $f(w_\lambda) \in N_\lambda$, that is, the map $f \mapsto f(w_\lambda)$ is well-defined. Now, to show that $f \mapsto f(w_\lambda)$ is a homomorphism of \mathbf{A}_λ -modules, notice that

$$(u \cdot f)(w_\lambda) = f(w_\lambda \cdot u) = f(uw_\lambda) = u(f(w_\lambda)) \quad \text{for all } u \in \mathbf{A}_\lambda.$$

To show that the map $f \mapsto f(w_\lambda)$ is injective, recall from Proposition 5.6 that $W_A(\lambda)$ is generated as a left $\mathbf{U}(\mathfrak{g} \otimes A)$ -module, by w_λ . Since f is a homomorphism of $\mathbf{U}(\mathfrak{g} \otimes A)$ -modules, f is thus uniquely determined by $f(w_\lambda)$.

To finish the proof, we will show that the map $f \mapsto f(w_\lambda)$ is surjective. Let $n \in N_\lambda$. Recall that N is an object of \mathcal{C}_A^λ , so $(\mathfrak{n}^+ \otimes A)n = 0$. Moreover, by Lemma 6.2, we also have $\text{Ann}_{\mathfrak{h} \otimes A}(w_\lambda)n = 0$. Furthermore, since N is a finitely-semisimple \mathfrak{r} -module, the \mathfrak{r} -submodule $\mathbf{U}(\mathfrak{r})n \subseteq N$ is finite dimensional. By the representation theory of finite-dimensional semisimple Lie algebras, we thus

have that $(x_\alpha^-)^{\lambda(h_\alpha)+1}n = 0$ for all $\alpha \in \Delta_{\mathfrak{r}}$. Hence, by Proposition 5.6, there exists a unique homomorphism of $\mathfrak{g} \otimes A$ -modules $f_n: W_A(\lambda) \rightarrow N$ satisfying $f_n(w_\lambda) = n$. The result follows. \square

Given $\lambda \in X^+$ and an object M of \mathcal{C}_A^λ , consider M_λ as an \mathbf{A}_λ -module. Given $\pi \in \text{Hom}_{\mathcal{C}_A^\lambda}(V, V')$, the restriction of π to V_λ induces a homomorphism of \mathbf{A}_λ -modules $\pi_\lambda: V_\lambda \rightarrow V'_\lambda$. We can thus define a functor

$$(6.3) \quad \mathbf{R}_A^\lambda: \mathcal{C}_A^\lambda \rightarrow \mathbf{A}_\lambda\text{-mod}, \quad \mathbf{R}_A^\lambda V = V_\lambda, \quad \mathbf{R}_A^\lambda(\pi) = \pi_\lambda.$$

Notice that \mathbf{R}_A^λ is an exact functor, since every object of \mathcal{C}_A^λ is a finitely-semisimple \mathfrak{r} -module, and thus a direct sum of its \mathfrak{h} -weight spaces, and every morphism of \mathcal{C}_A^λ preserves these weight spaces.

In the non-super setting, the next proposition was proved in [CFK10, Proposition 5] (in the untwisted case) and in [FMS15, Proposition 4.8] (in the twisted case).

Proposition 6.5. *Let $\lambda \in X^+$.*

- (a) *For every \mathbf{A}_λ -module M , there is an isomorphism of \mathbf{A}_λ -modules $\mathbf{R}_A^\lambda \mathbf{W}_A^\lambda M \cong M$ that is functorial in M .*
- (b) *$\mathbf{W}_A^\lambda: \mathbf{A}_\lambda\text{-mod} \rightarrow \mathcal{C}_A^\lambda$ is left adjoint to $\mathbf{R}_A^\lambda: \mathcal{C}_A^\lambda \rightarrow \mathbf{A}_\lambda\text{-mod}$.*
- (c) *If M is a projective \mathbf{A}_λ -module, then $\mathbf{W}_A^\lambda M$ is a projective object in \mathcal{C}_A^λ .*

Proof. First recall from Definition 6.3, (6.2) and (6.3) that

$$\mathbf{R}_A^\lambda \mathbf{W}_A^\lambda M = \mathbf{R}_A^\lambda (W_A(\lambda) \otimes_{\mathbf{A}_\lambda} M) = (W_A(\lambda) \otimes_{\mathbf{A}_\lambda} M)_\lambda = W_A(\lambda)_\lambda \otimes_{\mathbf{A}_\lambda} M.$$

Now, by Remark 6.1, there is an isomorphism of \mathbf{A}_λ -modules $\phi: \mathbf{A}_\lambda \rightarrow W_A(\lambda)_\lambda$; namely, $\phi(u) = uw_\lambda$ for all $u \in \mathbf{A}_\lambda$. Thus $\phi \otimes \text{id}_M: \mathbf{A}_\lambda \otimes_{\mathbf{A}_\lambda} M \rightarrow W_A(\lambda)_\lambda \otimes_{\mathbf{A}_\lambda} M$ is an isomorphism of \mathbf{A}_λ -modules that is functorial in M . Finally, notice that the unique homomorphism of \mathbf{A}_λ -modules satisfying

$$\mathbf{A}_\lambda \otimes_{\mathbf{A}_\lambda} M \rightarrow M, \quad u \otimes m \mapsto um, \quad \text{for all } u \in \mathbf{A}_\lambda \text{ and } m \in M,$$

is in fact an isomorphism that is functorial in M . Hence the result of part (a) follows.

To prove part (b), let M be an \mathbf{A}_λ -module and N be an object of \mathcal{C}_A^λ . By Definition 6.3, the tensor-hom adjunction, and Lemma 6.4, we have

$$\begin{aligned} \text{Hom}_{\mathcal{C}_A^\lambda}(\mathbf{W}_A^\lambda M, N) &= \text{Hom}_{\mathbf{U}(\mathfrak{g} \otimes A)}(W_A(\lambda) \otimes_{\mathbf{A}_\lambda} M, N) \\ &\cong \text{Hom}_{\mathbf{A}_\lambda}(M, \text{Hom}_{\mathbf{U}(\mathfrak{g} \otimes A)}(W_A(\lambda), N)) \\ &\cong \text{Hom}_{\mathbf{A}_\lambda}(M, N_\lambda) \\ &= \text{Hom}_{\mathbf{A}_\lambda}(M, \mathbf{R}_A^\lambda N), \end{aligned}$$

where the isomorphisms are functorial in M and N .

Part (c) follows from part (b), since \mathbf{W}_A^λ is left adjoint to an exact functor. \square

The following result follows from Proposition 6.5.

Corollary 6.6. *For each $\lambda \in X^+$, the module $W_A(\lambda)$ is projective in \mathcal{C}_A^λ and the module $K(\lambda)$ is projective in $\mathcal{C}_\mathbb{k}^\lambda$. Moreover, there is an isomorphism of algebras $\text{Hom}_{\mathcal{C}_A^\lambda}(W_A(\lambda), W_A(\lambda)) \cong \mathbf{A}_\lambda$.*

Proof. The first statement follows from Proposition 6.5(c) and the fact that \mathbf{A}_λ is a free (therefore projective) \mathbf{A}_λ -module. The *moreover* statement follows from Lemma 6.4 and Remark 6.1. \square

Lemma 6.7. *An object $M \in \mathcal{C}_A^\lambda$ satisfies $M = \mathbf{U}(\mathfrak{g} \otimes A)M_\lambda$ if and only if, for each object N of \mathcal{C}_A^λ that satisfies $N_\lambda = 0$, we have $\text{Hom}_{\mathcal{C}_A^\lambda}(M, N) = 0$.*

Proof. First assume that, for each object N of \mathcal{C}_A^λ that satisfies $N_\lambda = 0$, we have $\text{Hom}_{\mathcal{C}_A^\lambda}(M, N) = 0$. Denote the submodule $\mathbf{U}(\mathfrak{g} \otimes A)M_\lambda \subseteq M$ by M' , and consider the short exact sequence

$$0 \rightarrow M' \hookrightarrow M \rightarrow M/M' \rightarrow 0.$$

Since \mathbf{R}_A^λ is an exact functor, $0 \rightarrow M'_\lambda \hookrightarrow M_\lambda \rightarrow (M/M')_\lambda \rightarrow 0$ is also an exact sequence. Now notice that by construction, $M'_\lambda = M_\lambda$. Hence $(M/M')_\lambda = 0$. This implies, by hypothesis, that $\text{Hom}_{\mathcal{C}_A^\lambda}(M, M/M') = 0$. Thus $M' = M$.

Now assume that $\mathbf{U}(\mathfrak{g} \otimes A)M_\lambda = M$ and let N be an object N of \mathcal{C}_A^λ that satisfies $N_\lambda = 0$. Given a homomorphism of $\mathfrak{g} \otimes A$ -modules $\phi: M \rightarrow N$, consider the short exact sequence

$$0 \rightarrow \ker \phi \rightarrow M \xrightarrow{\phi} \text{im } \phi \rightarrow 0.$$

Since \mathbf{R}_A^λ is an exact functor, $0 \rightarrow (\ker \phi)_\lambda \rightarrow M_\lambda \xrightarrow{\phi_\lambda} (\text{im } \phi)_\lambda \rightarrow 0$ is also an exact sequence. Since N_λ is assumed to be zero, we have that $(\text{im } \phi)_\lambda = 0$ and $\phi_\lambda = 0$. Since $M = \mathbf{U}(\mathfrak{g} \otimes A)M_\lambda$, the homomorphism ϕ is uniquely determined by ϕ_λ . Thus $\phi = 0$, and the result follows. \square

Remark 6.8. Let V be an \mathbf{A}_λ -module. First notice that, by Proposition 6.5(a), we have $\mathbf{W}_A^\lambda V \cong \mathbf{W}_A^\lambda \mathbf{R}_A^\lambda \mathbf{W}_A^\lambda V$. Moreover, notice that $\mathbf{W}_A^\lambda V = \mathbf{U}(\mathfrak{g} \otimes A)(\mathbf{W}_A^\lambda V)_\lambda$. Thus, by Lemma 6.7, for every object N of \mathcal{C}_A^λ that satisfies $N_\lambda = 0$, we have $\text{Hom}_{\mathcal{C}_A^\lambda}(\mathbf{W}_A^\lambda V, N) = 0$. This observation will be used in the proofs of the next two results.

Recall from Corollary 5.4 that the category \mathcal{C}_A^λ has enough projectives. Thus, for $n \in \mathbb{N}$ and objects M, N of \mathcal{C}_A^λ , we can consider the group $\text{Ext}_{\mathcal{C}_A^\lambda}^n(M, N)$. In the non-super setting, the next result was proved in [CFK10, Theorem 1] and [FMS15, Theorem 4.10], for untwisted and twisted cases, respectively.

Theorem 6.9. *Let M be an object of \mathcal{C}_A^λ . Then $M \cong \mathbf{W}_A^\lambda \mathbf{R}_A^\lambda M$ if and only if, for each object N of \mathcal{C}_A^λ that satisfies $N_\lambda = 0$, we have*

$$\text{Hom}_{\mathcal{C}_A^\lambda}(M, N) = \text{Ext}_{\mathcal{C}_A^\lambda}^1(M, N) = 0.$$

Proof. First assume that M is an object of \mathcal{C}_A^λ such that, for each $N \in \mathcal{C}_A^\lambda$ that satisfies $N_\lambda = 0$, we have $\text{Hom}_{\mathcal{C}_A^\lambda}(M, N) = \text{Ext}_{\mathcal{C}_A^\lambda}^1(M, N) = 0$. Using Proposition 5.6, we see that there is a unique homomorphism of $\mathfrak{g} \otimes A$ -modules $\epsilon_M: \mathbf{W}_A^\lambda \mathbf{R}_A^\lambda M \rightarrow M$ satisfying

$$\epsilon_M(uw_\lambda \otimes m_\lambda) = um_\lambda \quad \text{for all } u \in \mathbf{U}(\mathfrak{g} \otimes A) \text{ and } m_\lambda \in M_\lambda.$$

Since $\text{Hom}_{\mathcal{C}_A^\lambda}(M, N) = 0$ for every object N of \mathcal{C}_A^λ that satisfies $N_\lambda = 0$, by Lemma 6.7, we have that $M = \mathbf{U}(\mathfrak{g} \otimes A)M_\lambda$. Hence, the homomorphism ϵ_M is surjective, and we have a short exact sequence

$$0 \rightarrow \ker \epsilon_M \rightarrow \mathbf{W}_A^\lambda \mathbf{R}_A^\lambda M \rightarrow M \rightarrow 0.$$

Consider the associated long exact sequence on $\text{Ext}_{\mathcal{C}_A^\lambda}^\bullet(-, \ker \epsilon_M)$. In particular, we have:

$$\cdots \rightarrow \text{Hom}_{\mathcal{C}_A^\lambda}(\mathbf{W}_A^\lambda \mathbf{R}_A^\lambda M, \ker \epsilon_M) \rightarrow \text{Hom}_{\mathcal{C}_A^\lambda}(\ker \epsilon_M, \ker \epsilon_M) \rightarrow \text{Ext}_{\mathcal{C}_A^\lambda}^1(M, \ker \epsilon_M) \rightarrow \cdots$$

Now, notice that $(\ker \epsilon_M)_\lambda = 0$, as ϵ_M maps $(\mathbf{W}_A^\lambda \mathbf{R}_A^\lambda M)_\lambda$ isomorphically onto M_λ . This implies by Remark 6.8, that $\text{Hom}_{\mathcal{C}_A^\lambda}(\mathbf{W}_A^\lambda \mathbf{R}_A^\lambda M, \ker \epsilon_M) = 0$, and by hypothesis, that $\text{Ext}_{\mathcal{C}_A^\lambda}^1(M, \ker \epsilon_M) = 0$. Hence, $\text{Hom}_{\mathcal{C}_A^\lambda}(\ker \epsilon_M, \ker \epsilon_M) = 0$. Thus $\ker \epsilon_M = 0$, and ϵ_M is an isomorphism between $\mathbf{W}_A^\lambda \mathbf{R}_A^\lambda M$ and M .

Now, to prove the converse, assume that $M \cong \mathbf{W}_A^\lambda \mathbf{R}_A^\lambda M$. It follows from Remark 6.8 that, for each object N of \mathcal{C}_A^λ that satisfies $N_\lambda = 0$, we have $\mathrm{Hom}_{\mathcal{C}_A^\lambda}(M, N) = 0$. Now, recall that the category $\mathbf{A}_\lambda\text{-mod}$ has enough projectives, that is, there exist a projective \mathbf{A}_λ -module P and a surjective homomorphism of \mathbf{A}_λ -modules $\pi: P \rightarrow M_\lambda$. Since the functor \mathbf{W}_A^λ is right exact, $\mathbf{W}_A^\lambda \pi$ is an even homomorphism, and $M \cong \mathbf{W}_A^\lambda \mathbf{R}_A^\lambda M$, we have a short exact sequence of $\mathfrak{g} \otimes A$ -modules

$$0 \rightarrow \ker(\mathbf{W}_A^\lambda \pi) \rightarrow \mathbf{W}_A^\lambda P \xrightarrow{\mathbf{W}_A^\lambda \pi} M \rightarrow 0,$$

where $\ker(\mathbf{W}_A^\lambda \pi)$ is the image of $\mathbf{W}_A^\lambda(\ker \pi)$ inside $\mathbf{W}_A^\lambda P$. Let N be an object of \mathcal{C}_A^λ that satisfies $N_\lambda = 0$, and consider the associated long exact sequence on $\mathrm{Ext}_{\mathcal{C}_A^\lambda}^\bullet(-, N)$. In particular, we have:

$$\cdots \rightarrow \mathrm{Hom}_{\mathcal{C}_A^\lambda}(\ker(\mathbf{W}_A^\lambda \pi), N) \rightarrow \mathrm{Ext}_{\mathcal{C}_A^\lambda}^1(M, N) \rightarrow \mathrm{Ext}_{\mathcal{C}_A^\lambda}^1(\mathbf{W}_A^\lambda P, N) \rightarrow \cdots$$

By Proposition 6.5(c), $\mathbf{W}_A^\lambda P$ is a projective object in \mathcal{C}_A^λ . Hence, $\mathrm{Ext}_{\mathcal{C}_A^\lambda}^1(\mathbf{W}_A^\lambda P, N) = 0$. Since $\ker(\mathbf{W}_A^\lambda \pi)$ is the image of $\mathbf{W}_A^\lambda(\ker \pi)$ inside $\mathbf{W}_A^\lambda P$, we have that $\ker(\mathbf{W}_A^\lambda \pi) = \mathbf{U}(\mathfrak{g} \otimes A) \ker(\mathbf{W}_A^\lambda \pi)_\lambda$. Hence, by Remark 6.8, we also have that $\mathrm{Hom}_{\mathcal{C}_A^\lambda}(\ker(\mathbf{W}_A^\lambda \pi), N) = 0$. The result follows. \square

Recall that $\mathbf{W}_A^\lambda: \mathbf{A}_\lambda\text{-mod} \rightarrow \mathcal{C}_A^\lambda$ is a right exact functor, that $\mathbf{A}_\lambda\text{-mod}$ has enough projectives, and that projective \mathbf{A}_λ -modules are left acyclic for \mathbf{W}_A^λ . Thus, we can consider the left derived functors $L_n \mathbf{W}_A^\lambda$, $n \geq 0$. In particular, $L_0 \mathbf{W}_A^\lambda = \mathbf{W}_A^\lambda$ and \mathbf{W}_A^λ is exact if and only if $L_1 \mathbf{W}_A^\lambda = 0$.

The next result generalizes [CFK10, Corollary 3] to the super setting.

Corollary 6.10. *The functor \mathbf{W}_A^λ is exact if and only if, for each object N of \mathcal{C}_A^λ that satisfies $N_\lambda = 0$, we have*

$$(6.4) \quad \mathrm{Ext}_{\mathcal{C}_A^\lambda}^2(\mathbf{W}_A^\lambda -, N) = 0.$$

Proof. Assume that $\mathrm{Ext}_{\mathcal{C}_A^\lambda}^2(\mathbf{W}_A^\lambda -, N) = 0$, and let V be an \mathbf{A}_λ -module. Recall that the category $\mathbf{A}_\lambda\text{-mod}$ has enough projectives, that is, there exist a projective \mathbf{A}_λ -module P and an exact sequence of \mathbf{A}_λ -modules $0 \rightarrow \ker \pi \xrightarrow{i} P \xrightarrow{\pi} V \rightarrow 0$. Thus, by the definition of $L_1 \mathbf{W}_A^\lambda$, we have an exact sequence of $\mathfrak{g} \otimes A$ -modules

$$0 \rightarrow L_1 \mathbf{W}_A^\lambda(V) \rightarrow \mathbf{W}_A^\lambda(\ker \pi) \xrightarrow{\mathbf{W}_A^\lambda i} \mathbf{W}_A^\lambda P \xrightarrow{\mathbf{W}_A^\lambda \pi} \mathbf{W}_A^\lambda V \rightarrow 0.$$

Denote $\ker(\mathbf{W}_A^\lambda \pi) = \mathrm{im}(\mathbf{W}_A^\lambda i)$ by K , consider the associated short exact sequences:

$$(6.5) \quad 0 \rightarrow L_1 \mathbf{W}_A^\lambda(V) \rightarrow \mathbf{W}_A^\lambda(\ker \pi) \xrightarrow{\mathbf{W}_A^\lambda i} K \rightarrow 0,$$

$$(6.6) \quad 0 \rightarrow K \hookrightarrow \mathbf{W}_A^\lambda P \xrightarrow{\mathbf{W}_A^\lambda \pi} \mathbf{W}_A^\lambda V \rightarrow 0,$$

and notice, by applying the exact functor \mathbf{R}_A^λ to (6.6), that $K_\lambda \cong \ker \pi$.

Let N be an object of \mathcal{C}_A^λ that satisfies $N_\lambda = 0$. The long exact sequence on $\mathrm{Ext}_{\mathcal{C}_A^\lambda}^\bullet(-, N)$ associated to (6.6) gives, in particular,

$$\begin{aligned} \cdots \rightarrow \mathrm{Hom}_{\mathcal{C}_A^\lambda}(\mathbf{W}_A^\lambda P, N) &\rightarrow \mathrm{Hom}_{\mathcal{C}_A^\lambda}(K, N) \rightarrow \mathrm{Ext}_{\mathcal{C}_A^\lambda}^1(\mathbf{W}_A^\lambda V, N) \rightarrow \\ &\rightarrow \mathrm{Ext}_{\mathcal{C}_A^\lambda}^1(\mathbf{W}_A^\lambda P, N) \rightarrow \mathrm{Ext}_{\mathcal{C}_A^\lambda}^1(K, N) \rightarrow \mathrm{Ext}_{\mathcal{C}_A^\lambda}^2(\mathbf{W}_A^\lambda V, N) \rightarrow \cdots \end{aligned}$$

By Lemma 6.7 and Theorem 6.9, we have that $\mathrm{Hom}_{\mathcal{C}_A^\lambda}(\mathbf{W}_A^\lambda P, N) = \mathrm{Ext}_{\mathcal{C}_A^\lambda}^1(\mathbf{W}_A^\lambda V, N) = 0$. Hence $\mathrm{Hom}_{\mathcal{C}_A^\lambda}(K, N) = 0$. By Proposition 6.5(c) and the hypothesis, we have that $\mathrm{Ext}_{\mathcal{C}_A^\lambda}^1(\mathbf{W}_A^\lambda P, N) =$

$\text{Ext}_{\mathcal{C}_A^\lambda}^2(\mathbf{W}_A^\lambda V, N) = 0$. Hence, $\text{Ext}_{\mathcal{C}_A^\lambda}^1(K, N) = 0$. Thus, by Theorem 6.9, we have that

$$K \cong \mathbf{W}_A^\lambda \mathbf{R}_A^\lambda K \cong \mathbf{W}_A^\lambda(\ker \pi).$$

That is, $\mathbf{W}_A^\lambda i: \mathbf{W}_A^\lambda(\ker \pi) \rightarrow K$ is an isomorphism, and using (6.5), we see that $L_1 \mathbf{W}_A^\lambda(V) = 0$.

Now, to prove the converse, assume that \mathbf{W}_A^λ is exact, and let V be an \mathbf{A}_λ -module. Since the category $\mathbf{A}_\lambda\text{-mod}$ has enough projectives, there exist a projective \mathbf{A}_λ -module P and a surjective homomorphism of \mathbf{A}_λ -modules $\pi: P \rightarrow V$. Since the functor \mathbf{W}_A^λ is assumed to be exact and $\mathbf{W}_A^\lambda \pi$ is an even homomorphism we have a short exact sequence of $\mathfrak{g} \otimes A$ -modules

$$0 \rightarrow \mathbf{W}_A^\lambda(\ker \pi) \rightarrow \mathbf{W}_A^\lambda P \rightarrow \mathbf{W}_A^\lambda V \rightarrow 0.$$

Now let N be an object of \mathcal{C}_A^λ that satisfies $N_\lambda = 0$, and consider the associated long exact sequence on $\text{Ext}_{\mathcal{C}_A^\lambda}^\bullet(-, N)$. In particular, we have:

$$\cdots \rightarrow \text{Ext}_{\mathcal{C}_A^\lambda}^1(\mathbf{W}_A^\lambda(\ker \pi), N) \rightarrow \text{Ext}_{\mathcal{C}_A^\lambda}^2(\mathbf{W}_A^\lambda V, N) \rightarrow \text{Ext}_{\mathcal{C}_A^\lambda}^2(\mathbf{W}_A^\lambda P, N) \rightarrow \cdots$$

By Theorem 6.9, $\text{Ext}_{\mathcal{C}_A^\lambda}^1(\mathbf{W}_A^\lambda(\ker \pi), N) = 0$, and by Proposition 6.5(c), $\text{Ext}_{\mathcal{C}_A^\lambda}^2(\mathbf{W}_A^\lambda P, N) = 0$. Thus, as we wanted to prove, $\text{Ext}_{\mathcal{C}_A^\lambda}^2(\mathbf{W}_A^\lambda V, N) = 0$. \square

7. THE STRUCTURE OF $W_A(\lambda)$ AS A RIGHT \mathbf{A}_λ -MODULE

Recall that \mathfrak{g} is assumed to be either $\mathfrak{sl}(n, n)$ with $n \geq 2$, or a finite-dimensional simple Lie superalgebra not of type $\tilde{S}(n)$ or $\mathfrak{q}(n)$, and A is assumed to be an associative commutative \mathbb{k} -algebra with unit. For the remainder of the paper, we will also assume that A is finitely generated.

Recall from (2.1) that \mathfrak{r} is a finite-dimensional reductive Lie algebra with Cartan subalgebra \mathfrak{h} , that, for every $\alpha \in R_\tau^+$, the subalgebra $\mathfrak{sl}_\alpha \subseteq \mathfrak{r}$, which is generated by $\{x_\alpha^-, h_\alpha, x_\alpha\}$, is isomorphic to $\mathfrak{sl}(2)$, and recall from Definition 5.5 that, for every $\lambda \in X^+$, the global Weyl module $W_A(\lambda)$ is generated by its highest-weight vector $w_\lambda \in W_A(\lambda)_\lambda$.

Lemma 7.1. *If $\lambda \in X^+$ and $\alpha \in R_\tau^+$, then $(x_\alpha^-)^{\lambda(h_\alpha)+1} w_\lambda = 0$.*

Proof. Recall that the global Weyl module $W_A(\lambda)$ is a direct sum of its irreducible finite-dimensional \mathfrak{r} -submodules. Thus all the weights of $W_A(\lambda)$ are invariant under the action of the Weyl group of \mathfrak{r} . Also recall that $W_A(\lambda)$ is a highest-weight module of highest weight λ . In particular, if we denote by s_α the reflection in the Weyl group of \mathfrak{r} associated to the root α , then $W_A(\lambda)_{s_\alpha(\lambda+\alpha)} = 0$. Now notice that $(x_\alpha^-)^{\lambda(h_\alpha)+1} w_\lambda$ has weight $\lambda - (\lambda(h_\alpha) + 1)\alpha = s_\alpha(\lambda + \alpha)$. Thus $(x_\alpha^-)^{\lambda(h_\alpha)+1} w_\lambda = 0$. \square

Given $a \in A$ and $\alpha \in R_\tau^+$, define a power series in an indeterminate u and with coefficients in $\mathbf{U}(\mathfrak{h} \otimes A)$ as follows:

$$(7.1) \quad p(a, \alpha) = \exp \left(- \sum_{i=1}^{\infty} \frac{h_\alpha \otimes a^i}{i} u^i \right).$$

For $i \geq 0$, let $p(a, \alpha)_i$ denote the coefficient of u^i in $p(a, \alpha)$, and notice that $p(a, \alpha)_0 = 1$.

The following lemma was proved by Garland.

Lemma 7.2 ([Gar78, Lemma 7.5]). *Let $m \in \mathbb{N}$, $a \in A$ and $\alpha \in R_\tau^+$. Then*

$$(x_\alpha \otimes a)^m (x_\alpha^-)^{m+1} - (-1)^m \sum_{i=0}^m (x_\alpha^- \otimes a^{m-i}) p(a, \alpha)_i \in \mathbf{U}(\mathfrak{sl}_\alpha \otimes A)(\mathfrak{g}_\alpha \otimes A),$$

where $\mathbf{U}(\mathfrak{sl}_\alpha \otimes A)(\mathfrak{g}_\alpha \otimes A)$ denotes the left ideal of $\mathbf{U}(\mathfrak{sl}_\alpha \otimes A)$ generated by $\mathfrak{g}_\alpha \otimes A = \mathbb{k}x_\alpha \otimes A$.

Analogues of the previous lemma have also been proved in [BC14, Proposition 4.1.2] when \mathfrak{g} is any basic classical Lie superalgebra, and in [BC16, Proposition 4.1.2] when \mathfrak{g} is of Cartan type.

Lemma 7.3. *Let $\lambda \in X^+$, $\alpha \in R_{\mathfrak{t}}^+$, and $a_1, \dots, a_t \in A$. Then, for every $m_1, \dots, m_t \in \mathbb{N}$, we have:*

$$(7.2) \quad (x_{\alpha}^{-} \otimes a_1^{m_1} \cdots a_t^{m_t})w_{\lambda} \in \text{span}_{\mathbb{K}} \{(x_{\alpha}^{-} \otimes a_1^{\ell_1} \cdots a_t^{\ell_t})w_{\lambda} \mathbf{A}_{\lambda} \mid 0 \leq \ell_1, \dots, \ell_t < \lambda(h_{\alpha})\}.$$

In particular, $(\mathfrak{r} \otimes A)w_{\lambda}$ is a finitely-generated right \mathbf{A}_{λ} -module.

Proof. We will use induction on t . First assume that $t = 1$, and fix $a \in A$. From Lemma 7.1, the first relation in (5.6), and Lemma 7.2, we have:

$$0 = (x_{\alpha} \otimes a)^m (x_{\alpha}^{-})^{m+1} w_{\lambda} = \sum_{i=0}^m (-1)^m (x_{\alpha}^{-} \otimes a^{m-i}) p(a, \alpha)_i w_{\lambda} \quad \text{for all } m \geq \lambda(h_{\alpha}).$$

Thus, using the fact that $p(a, \alpha)_0 = 1$ and induction on m , we conclude that

$$(x_{\alpha}^{-} \otimes a^m)w_{\lambda} \in \text{span}_{\mathbb{K}} \{(x_{\alpha}^{-} \otimes a^{\ell})w_{\lambda} \mathbf{A}_{\lambda} \mid 0 \leq \ell < \lambda(h_{\alpha})\} \quad \text{for all } m \in \mathbb{N}.$$

This proves the case $t = 1$.

Now, let $s > 1$, assume that (7.2) holds for all $t \leq s$, and fix $a_1, \dots, a_{s+1} \in A$. Since

$$(7.3) \quad [x_{\alpha}^{-}, h_{\alpha}] = 2x_{\alpha}^{-} \quad \text{and} \quad A \text{ is assumed to be commutative,}$$

we have:

$$2(x_{\alpha}^{-} \otimes a_1^{m_1} \cdots a_{s+1}^{m_{s+1}})w_{\lambda} = (x_{\alpha}^{-} \otimes a_1^{m_1} \cdots a_s^{m_s})(h_{\alpha} \otimes a_{s+1}^{m_{s+1}})w_{\lambda} - (h_{\alpha} \otimes a_{s+1}^{m_{s+1}})(x_{\alpha}^{-} \otimes a_1^{m_1} \cdots a_s^{m_s})w_{\lambda},$$

for all $m_1, \dots, m_{s+1} \in \mathbb{N}$. Thus, using the fact that $(h_{\alpha} \otimes a_{s+1}^{m_{s+1}})w_{\lambda} \in w_{\lambda} \mathbf{A}_{\lambda}$ and the induction hypothesis (for $t = s$), we see that

$$\begin{aligned} (x_{\alpha}^{-} \otimes a_1^{m_1} \cdots a_{s+1}^{m_{s+1}})w_{\lambda} &\in \text{span}_{\mathbb{K}} \{(x_{\alpha}^{-} \otimes a_1^{\ell_1} \cdots a_s^{\ell_s})w_{\lambda} \mathbf{A}_{\lambda} \mid 0 \leq \ell_i < \lambda(h_{\alpha}), i = 1, \dots, s\} \\ &\quad + \text{span}_{\mathbb{K}} \{(h_{\alpha} \otimes a_{s+1}^{m_{s+1}})(x_{\alpha}^{-} \otimes a_1^{\ell_1} \cdots a_s^{\ell_s})w_{\lambda} \mid 0 \leq \ell_i < \lambda(h_{\alpha}), i = 1, \dots, s\}, \end{aligned}$$

for all $m_1, \dots, m_{s+1} \in \mathbb{N}$. Using (7.3) again, we have:

$$\begin{aligned} (h_{\alpha} \otimes a_{s+1}^{m_{s+1}})(x_{\alpha}^{-} \otimes a_1^{\ell_1} \cdots a_s^{\ell_s})w_{\lambda} &= (x_{\alpha}^{-} \otimes a_1^{\ell_1} \cdots a_s^{\ell_s})(h_{\alpha} \otimes a_{s+1}^{m_{s+1}})w_{\lambda} - 2(x_{\alpha}^{-} \otimes a_1^{\ell_1} \cdots a_s^{\ell_s} a_{s+1}^{m_{s+1}})w_{\lambda} \\ &= (x_{\alpha}^{-} \otimes a_1^{\ell_1} \cdots a_s^{\ell_s})(h_{\alpha} \otimes a_{s+1}^{m_{s+1}})w_{\lambda} + (h_{\alpha} \otimes a_1^{\ell_1})(x_{\alpha}^{-} \otimes a_2^{\ell_2} \cdots a_s^{\ell_s} a_{s+1}^{m_{s+1}})w_{\lambda} \\ &\quad - (x_{\alpha}^{-} \otimes a_2^{\ell_2} \cdots a_s^{\ell_s} a_{s+1}^{m_{s+1}})(h_{\alpha} \otimes a_1^{\ell_1})w_{\lambda}, \end{aligned}$$

for all $0 \leq \ell_1, \dots, \ell_s < \lambda(h_{\alpha})$ and $m_{s+1} \in \mathbb{N}$. Thus, using the induction hypothesis again (for $t = s$) on $(h_{\alpha} \otimes a_1^{\ell_1})(x_{\alpha}^{-} \otimes a_2^{\ell_2} \cdots a_s^{\ell_s} a_{s+1}^{m_{s+1}})w_{\lambda}$ and $(x_{\alpha}^{-} \otimes a_2^{\ell_2} \cdots a_s^{\ell_s} a_{s+1}^{m_{s+1}})(h_{\alpha} \otimes a_1^{\ell_1})w_{\lambda}$, we see that

$$\begin{aligned} (h_{\alpha} \otimes a_{s+1}^{m_{s+1}})(x_{\alpha}^{-} \otimes a_1^{\ell_1} \cdots a_s^{\ell_s})w_{\lambda} &\in \text{span}_{\mathbb{K}} \{(x_{\alpha}^{-} \otimes a_1^{k_1} \cdots a_{s+1}^{k_{s+1}})w_{\lambda} \mathbf{A}_{\lambda} \mid 0 \leq k_i < \lambda(h_{\alpha}), i = 1, \dots, s+1\} \\ &\quad + \text{span}_{\mathbb{K}} \{(h_{\alpha} \otimes a_1^{\ell_1})(x_{\alpha}^{-} \otimes a_2^{k_2} \cdots a_{s+1}^{k_{s+1}})w_{\lambda} \mid 0 \leq k_i < \lambda(h_{\alpha}), i = 2, \dots, s+1\}, \end{aligned}$$

for all $0 \leq \ell_1, \dots, \ell_s < \lambda(h_{\alpha})$ and $m_{s+1} \in \mathbb{N}$. Finally, using (7.3) again, we have:

$$\begin{aligned} (h_{\alpha} \otimes a_1^{\ell_1})(x_{\alpha}^{-} \otimes a_2^{k_2} \cdots a_{s+1}^{k_{s+1}})w_{\lambda} &= (x_{\alpha}^{-} \otimes a_2^{k_2} \cdots a_{s+1}^{k_{s+1}})(h_{\alpha} \otimes a_1^{\ell_1})w_{\lambda} - 2(x_{\alpha}^{-} \otimes a_1^{\ell_1} a_2^{k_2} \cdots a_{s+1}^{k_{s+1}})w_{\lambda} \\ &\in \text{span}_{\mathbb{K}} \{(x_{\alpha}^{-} \otimes a_1^{n_1} \cdots a_{s+1}^{n_{s+1}})w_{\lambda} \mathbf{A}_{\lambda} \mid 0 \leq n_i < \lambda(h_{\alpha}), i = 1, \dots, s+1\}, \end{aligned}$$

for all $0 \leq \ell_1, k_2, \dots, k_{s+1} < \lambda(h_{\alpha})$. Hence, (7.2) follows.

In particular, using (7.2) and the assumptions that A is finitely generated and \mathfrak{r} is a finite-dimensional Lie algebra, we conclude that $(\mathfrak{r} \otimes A)w_\lambda$ is a finitely-generated right \mathbf{A}_λ -module. \square

Lemma 7.4. *Let $\lambda \in X^+$, $\alpha \in R_\tau^+$, $x_1, \dots, x_k \in \mathfrak{n}^+$ and $a_1, \dots, a_t \in A$. Then, for all $m_1, \dots, m_t \in \mathbb{N}$, the element $([x_1, [x_2, \dots [x_k, x_\alpha^-] \dots]] \otimes a_1^{m_1} \dots a_t^{m_t})w_\lambda$ is in*

$$\text{span}_{\mathbb{K}}\{([x_1, [x_2, \dots [x_k, x_\alpha^-] \dots]] \otimes a_1^{\ell_1} \dots a_t^{\ell_t})w_\lambda \mathbf{A}_\lambda \mid 0 \leq \ell_1, \dots, \ell_t < \lambda(h_\alpha)\}.$$

Proof. The proof is by induction on k . First assume that $k = 1$ and let $x \in \mathfrak{n}^+$. Using (7.2) and the first relation in (5.6), for all $m_1, \dots, m_t \in \mathbb{N}$, we have:

$$\begin{aligned} ([x, x_\alpha^-] \otimes a_1^{m_1} \dots a_t^{m_t})w_\lambda &= [x \otimes 1, x_\alpha^- \otimes a_1^{m_1} \dots a_t^{m_t}]w_\lambda \\ &= (x \otimes 1)(x_\alpha^- \otimes a_1^{m_1} \dots a_t^{m_t})w_\lambda \\ &\in \text{span}_{\mathbb{K}}\{([x, x_\alpha^-] \otimes a_1^{\ell_1} \dots a_t^{\ell_t})w_\lambda \mathbf{A}_\lambda \mid 0 \leq \ell_1, \dots, \ell_t < \lambda(h_\alpha)\}. \end{aligned}$$

This proves the case $k = 1$. Now assume $k > 1$ and let $x_1, \dots, x_k \in \mathfrak{n}^+$. Using the first relation in (5.6) and the induction hypothesis, for all $m_1, \dots, m_t \in \mathbb{N}$, we have:

$$\begin{aligned} ([x_1, [x_2, \dots [x_k, x_\alpha^-] \dots]] \otimes a_1^{m_1} \dots a_t^{m_t})w_\lambda &= [(x_1 \otimes 1), ([x_2, [x_3, \dots [x_k, x_\alpha^-] \dots]] \otimes a_1^{m_1} \dots a_t^{m_t})]w_\lambda \\ &= (x_1 \otimes 1)([x_2, [x_3, \dots [x_k, x_\alpha^-] \dots]] \otimes a_1^{m_1} \dots a_t^{m_t})w_\lambda \\ &\in \{([x_1, [x_2, \dots [x_k, x_\alpha^-] \dots]] \otimes a_1^{\ell_1} \dots a_t^{\ell_t})w_\lambda \mathbf{A}_\lambda \mid 0 \leq \ell_1, \dots, \ell_t < \lambda(h_\alpha)\}. \quad \square \end{aligned}$$

For the remainder of this section we will assume that \mathfrak{g} is a finite-dimensional simple Lie superalgebra not of type $\tilde{S}(n)$, $\mathfrak{p}(n)$ or $\mathfrak{q}(n)$. Thus, by Theorem 3.6, we can fix a triangular decomposition of \mathfrak{g} that satisfies (C2).

Theorem 7.5. *Let \mathfrak{g} be a finite-dimensional simple Lie superalgebra not of type $\tilde{S}(n)$, $\mathfrak{p}(n)$ or $\mathfrak{q}(n)$. For all $\lambda \in X^+$, the global Weyl module $W_A(\lambda)$ is finitely generated as a right \mathbf{A}_λ -module*

Proof. Let $\mathbf{U}(\mathfrak{n}^- \otimes A) = \sum_{n \geq 0} \mathbf{U}_n(\mathfrak{n}^- \otimes A)$ be the filtration of $\mathbf{U}(\mathfrak{n}^- \otimes A)$ induced from the usual grading of the symmetric algebra $S(\mathfrak{n}^- \otimes A) = \bigoplus_{d \geq 0} S^d(\mathfrak{n}^- \otimes A)$. Now, notice that, since $W_A(\lambda)$ is a finitely-semisimple \mathfrak{r} -module, its set of weights is invariant under the action of the Weyl group of \mathfrak{r} . Moreover, since $W_A(\lambda)$ is a highest-weight module of highest weight λ and the Weyl group of \mathfrak{r} is finite, the set of weights of $W_A(\lambda)$ is also finite. Thus, there exists $n_0 \in \mathbb{N}_+$ such that

$$\mathbf{U}_n(\mathfrak{n}^- \otimes A)w_\lambda \mathbf{A}_\lambda = W_A(\lambda) \quad \text{for all } n \geq n_0.$$

We will show that, for every $n \geq 0$, $\mathbf{U}_n(\mathfrak{n}^- \otimes A)w_\lambda \mathbf{A}_\lambda$ is a finitely-generated \mathbf{A}_λ -module. First, denote by $-\theta$ the lowest root of \mathfrak{g} . Notice that, since we have fixed a triangular decomposition of \mathfrak{g} satisfying (C2), we have $\theta \in R_\tau^+$. Also notice that, since \mathfrak{g} is assumed to be finite dimensional, there exists $k_0 \in \mathbb{N}$ such that $[x_1, [x_2, \dots [x_k, x_\theta^-] \dots]] = 0$, for all $k > k_0$ and $x_1, \dots, x_k \in \mathfrak{n}^+$. Moreover, since \mathfrak{g} is assumed to be simple, and x_θ^- is a lowest-weight vector in the \mathfrak{g} -module \mathfrak{g} , we have

$$(7.4) \quad \mathfrak{n}^- \subseteq \text{span}_{\mathbb{K}}\{[x_1, [x_2, \dots [x_k, x_\theta^-] \dots]] \mid x_1, \dots, x_k \in \mathfrak{n}^+ \text{ and } 0 \leq k \leq k_0\}.$$

Now recall that A is assumed to be finitely generated and let a_1, a_2, \dots, a_t be generators of A . Denote by $\mathcal{B}_{\mathfrak{n}^-}$ a (finite) basis of \mathfrak{n}^- extracted from the right side of (7.4) and let $\mathcal{B}_{\mathfrak{n}^- \otimes A}$ be the (finite) set

$$\{y \otimes a_1^{\ell_1} \dots a_t^{\ell_t} \mid y \in \mathcal{B}_{\mathfrak{n}^-} \text{ and } 0 \leq \ell_1, \dots, \ell_t < \lambda(h_\theta)\}.$$

We will use induction to prove that, for every $n \in \mathbb{N}_+$,

$$\mathbf{U}_n(\mathfrak{n}^- \otimes A)w_\lambda \subseteq \text{span}_{\mathbb{K}}\{Y_1^{n_1} \cdots Y_t^{n_t} w_\lambda \mathbf{A}_\lambda \mid t \geq 0, Y_1, \dots, Y_t \in \mathcal{B}_{\mathfrak{n}^- \otimes A} \text{ and } n_1 + \cdots + n_t \leq n\}.$$

For $n = 1$, the result follows from Lemma 7.4 and the construction of $\mathcal{B}_{\mathfrak{n}^- \otimes A}$. Suppose now $n > 1$. Without loss of generality, let $u = u_1 u_{n-1}$ be a monomial, with $u_1 \in \mathbf{U}_1(\mathfrak{n}^- \otimes A)$ and $u_{n-1} \in \mathbf{U}_{n-1}(\mathfrak{n}^- \otimes A)$. By induction hypothesis, we have:

$$uw_\lambda = u_1 u_{n-1} w_\lambda \in \text{span}_{\mathbb{K}}\{u_1 Y_1^{n_1} \cdots Y_t^{n_t} w_\lambda \mathbf{A}_\lambda \mid t \geq 0, Y_1, \dots, Y_t \in \mathcal{B}_{\mathfrak{n}^- \otimes A}, n_1 + \cdots + n_t \leq n-1\}.$$

Let u' be an element in $\text{span}_{\mathbb{K}}\{Y_1^{n_1} \cdots Y_t^{n_t} w_\lambda \mathbf{A}_\lambda \mid t \geq 0, Y_1, \dots, Y_t \in \mathcal{B}_{\mathfrak{n}^- \otimes A}, n_1 + \cdots + n_t \leq n-1\}$, and without loss of generality assume that u_1 and u' are homogeneous. By induction hypothesis, we have:

$$\begin{aligned} u_1 u' w_\lambda &= [u_1, u'] w_\lambda + (-1)^{p(u_1)p(u')} u' u_1 w_\lambda \\ &\in \mathbf{U}_{n-1}(\mathfrak{n}^- \otimes A) w_\lambda \mathbf{A}_\lambda + \text{span}_{\mathbb{K}}\{u' Y w_\lambda \mathbf{A}_\lambda \mid Y \in \mathcal{B}_{\mathfrak{n}^- \otimes A}\} \\ &\subseteq \text{span}_{\mathbb{K}}\{Y_1^{n_1} \cdots Y_{t+1}^{n_{t+1}} w_\lambda \mathbf{A}_\lambda \mid t \geq 0, Y_1, \dots, Y_{t+1} \in \mathcal{B}_{\mathfrak{n}^- \otimes A} \text{ and } n_1 + \cdots + n_{t+1} \leq n\}. \end{aligned}$$

The result follows. \square

In the non-super setting, Theorem 7.5 was proved in [CFK10, Theorem 2(i)] for the untwisted case, and in [FMS15, Theorem 5.10] for the twisted case. Notice that in the non-super setting the analogues of Theorem 7.5 do not depend on the choice of the triangular decomposition of \mathfrak{g} .

Proposition 7.6. *Let \mathfrak{g} be a finite-dimensional simple Lie superalgebra not of type $\tilde{S}(n)$, $\mathfrak{p}(n)$ or $\mathfrak{q}(n)$. For all $\lambda \in X^+$, the algebra \mathbf{A}_λ is finitely generated.*

Proof. Since \mathbf{A}_λ is defined to be $\mathbf{U}(\mathfrak{h} \otimes A) / \text{Ann}_{\mathfrak{h} \otimes A}(w_\lambda)$, to prove that \mathbf{A}_λ is finitely generated is equivalent to proving that there exist finitely many elements $H_1, \dots, H_n \in \mathbf{U}(\mathfrak{h} \otimes A)$ such that

$$\mathbf{U}(\mathfrak{h} \otimes A) w_\lambda = \text{span}_{\mathbb{K}}\{H_1^{k_1} \cdots H_n^{k_n} w_\lambda \mid k_1, \dots, k_n \geq 0\}.$$

Moreover, since $\mathbf{U}(\mathfrak{h} \otimes A)$ is a commutative algebra generated by $\mathfrak{h} \otimes A$, this is equivalent to proving that

$$(7.5) \quad (\mathfrak{h} \otimes A) w_\lambda \subseteq \text{span}_{\mathbb{K}}\{H_1^{k_1} \cdots H_n^{k_n} w_\lambda \mid k_1, \dots, k_n \geq 0\}.$$

In order to prove (7.5), first recall that A is assumed to be finitely generated and let a_1, a_2, \dots, a_t be generators of A . Now denote by $-\theta$ the lowest root of \mathfrak{g} . Notice that, since we have fixed a triangular decomposition of \mathfrak{g} satisfying (C2), we have $\theta \in R_r^+$. Also notice that, since \mathfrak{g} is assumed to be finite dimensional, there exists $k_0 \in \mathbb{N}$ such that $[x_1, [x_2, \dots [x_k, x_\theta^-] \dots]] = 0$, for all $k > k_0$ and $x_1, \dots, x_k \in \mathfrak{n}^+$. Moreover, since \mathfrak{g} is assumed to be simple and x_θ^- is a lowest-weight vector in the \mathfrak{g} -module \mathfrak{g} , we have

$$\mathfrak{h} \otimes A \subseteq \text{span}_{\mathbb{K}}\{[x_1, [x_2, \dots [x_k, x_\theta^-] \dots]] \otimes a_1^{m_1} \cdots a_t^{m_t} \mid x_1, \dots, x_k \in \mathfrak{n}^+, 0 < k \leq k_0, 0 \leq m_1, \dots, m_t\}.$$

Using arguments similar to those used in the proof of Lemma 7.3, we see that for every $k \in \mathbb{N}_+$ and $x_1, \dots, x_k \in \mathfrak{n}^+$ such that $[x_1, [x_2, \dots [x_k, x_\theta^-] \dots]] \in \mathfrak{h}$, the element $([x_1, [x_2, \dots [x_k, x_\theta^-] \dots]] \otimes a_1^{m_1} \cdots a_t^{m_t}) w_\lambda$ is a linear combination of elements of the form

$$([x_1, [x_2, \dots [x_k, x_\theta^-] \dots]] \otimes a_1^{\ell_1} \cdots a_t^{\ell_t}) P(\theta, k_1, \dots, k_t) w_\lambda,$$

where $0 \leq \ell_1, \dots, \ell_t < \lambda(h_\theta)$, $0 \leq k_1, \dots, k_t \leq \lambda(h_\theta)$, and $P(\theta, k_1, \dots, k_t)$ is a finite product of elements of $\mathbf{U}(\mathfrak{h} \otimes A)$ of the form $(h_\theta \otimes a_1^{k_1} \cdots a_t^{k_t})$. Thus the result follows. \square

The next result follows directly from Theorem 7.5.

Corollary 7.7. *Let \mathfrak{g} be a finite-dimensional simple Lie superalgebra not of type $\tilde{S}(n)$, $\mathfrak{p}(n)$ or $\mathfrak{q}(n)$. If M is a finitely-generated \mathbf{A}_λ -module (resp. finite dimensional), then $\mathbf{W}_A^\lambda M$ is a finitely-generated $\mathfrak{g} \otimes A$ -module (resp. finite dimensional).*

8. TENSOR PRODUCTS OF GLOBAL WEYL MODULES

Recall that \mathfrak{g} is assumed to be either $\mathfrak{sl}(n, n)$ with $n \geq 2$, or a finite-dimensional simple Lie superalgebra not of type $\tilde{S}(n)$ or $\mathfrak{q}(n)$. In this section, we will also assume that A, B and C are associative, commutative \mathbb{k} -algebras with unit. Similar to the definition of \mathbf{A}_λ in Section 6, for each $\lambda \in X^+$, let

$$\mathbf{B}_\lambda = \mathbf{U}(\mathfrak{h} \otimes B) / \text{Ann}_{\mathfrak{h} \otimes B}(w_\lambda) \quad \text{and} \quad \mathbf{C}_\lambda = \mathbf{U}(\mathfrak{h} \otimes C) / \text{Ann}_{\mathfrak{h} \otimes C}(w_\lambda).$$

In this section we will describe the interaction between Weyl functors and tensor products, generalizing certain results proved in the non-super setting in [CFK10, §4].

Remark 8.1. (a) Every homomorphism of \mathbb{k} -algebras $\pi: C \rightarrow A$ induces a unique (even) homomorphism of Lie superalgebras (which we denote by the same symbol) $\pi: \mathfrak{g} \otimes C \rightarrow \mathfrak{g} \otimes A$ satisfying

$$\pi(x \otimes c) = x \otimes \pi(c) \quad \text{for all } x \in \mathfrak{g} \text{ and } c \in C.$$

This latter homomorphism induces an action of $\mathfrak{g} \otimes C$ on any $\mathfrak{g} \otimes A$ -module M via the pull-back. Let π^*M denote such a $\mathfrak{g} \otimes C$ -module.

- (b) Let $\lambda \in X^+$ and $\pi: C \rightarrow A$ be a homomorphism of \mathbb{k} -algebras. Using item (a), we see that π also induces a homomorphism of associative superalgebras (which we keep denoting by the same symbol), $\pi: \mathbf{U}(\mathfrak{g} \otimes C) \rightarrow \mathbf{U}(\mathfrak{g} \otimes A)$. Notice that by construction, $\pi(\mathfrak{n}^+ \otimes C) \subseteq \mathfrak{n}^+ \otimes A$, $\pi(h) - \lambda(h) = h - \lambda(h)$ for all $h \in \mathfrak{h}$, and $\pi(x_\alpha^-)^k = (x_\alpha^-)^k$ for all $\alpha \in \Delta_{\mathfrak{r}}$ and $k \geq 0$. Hence

$$\pi(\text{Ann}_{\mathfrak{g} \otimes C}(w_\lambda)) \subseteq \text{Ann}_{\mathfrak{g} \otimes A}(w_\lambda) \quad \text{and} \quad \pi(\text{Ann}_{\mathfrak{h} \otimes C}(w_\lambda)) \subseteq \text{Ann}_{\mathfrak{h} \otimes A}(w_\lambda).$$

Thus π induces a homomorphism of \mathbb{k} -algebras $\bar{\pi}: \mathbf{C}_\lambda \rightarrow \mathbf{A}_\lambda$, and every \mathbf{A}_λ -module V admits a structure of \mathbf{C}_λ -module via the pull-back along $\bar{\pi}$. Denote this \mathbf{C}_λ -module by $\bar{\pi}^*V$.

- (c) Let $\lambda, \mu \in X^+$ be such that $\lambda + \mu \in X^+$, and recall that the action of the superalgebra $\mathbf{U}(\mathfrak{g} \otimes C)$ on $W_A(\lambda) \otimes W_A(\mu)$ is induced by the comultiplication $\Delta: \mathbf{U}(\mathfrak{g} \otimes C) \rightarrow \mathbf{U}(\mathfrak{g} \otimes C) \otimes \mathbf{U}(\mathfrak{g} \otimes C)$. In particular, we have $x(w_\lambda \otimes w_\mu) = (xw_\lambda) \otimes w_\mu + w_\lambda \otimes (xw_\mu)$ for all $x \in \mathfrak{g} \otimes C$, and thus $w_\lambda \otimes w_\mu$ is a highest-weight vector in $W_A(\lambda) \otimes W_A(\mu)$. Hence, there exists a unique surjective homomorphism of $\mathfrak{g} \otimes A$ -modules $\xi: W_A(\lambda + \mu) \twoheadrightarrow W_A(\lambda) \otimes W_A(\mu)$ satisfying $\xi(w_{\lambda+\mu}) = w_\lambda \otimes w_\mu$. Now, notice that $\mathbf{R}_A^{\lambda+\mu}\xi$ is a surjective homomorphism of $\mathbf{U}(\mathfrak{h} \otimes C)$ -modules:

$$\xi_{\lambda+\mu}: \mathbf{U}(\mathfrak{h} \otimes C)w_{\lambda+\mu} \twoheadrightarrow \mathbf{U}(\mathfrak{h} \otimes C)w_\lambda \otimes \mathbf{U}(\mathfrak{h} \otimes C)w_\mu.$$

Moreover, since $\mathbf{U}(\mathfrak{h} \otimes C)w_\nu \cong \mathbf{C}_\nu$ for all $\nu \in X^+$, $\xi_{\lambda+\mu}$ induces a homomorphism of commutative \mathbb{k} -algebras $\Delta_{\lambda,\mu}: \mathbf{C}_{\lambda+\mu} \rightarrow \mathbf{C}_\lambda \otimes \mathbf{C}_\mu$. Thus, given a \mathbf{C}_λ -module M and a \mathbf{C}_μ -module N , their tensor product $M \otimes N$ admits a $\mathbf{C}_{\lambda+\mu}$ -module structure via the pull-back along $\Delta_{\lambda,\mu}$. Denote this $\mathbf{C}_{\lambda+\mu}$ -module by $\Delta_{\lambda,\mu}^*(M \otimes N)$.

Proposition 8.2. *Let $\lambda, \mu \in X^+$ be such that $\lambda + \mu \in X^+$, $C = A \oplus B$, and $\pi_A: C \twoheadrightarrow A$, $\pi_B: C \twoheadrightarrow B$ denote the canonical surjective homomorphisms of \mathbb{k} -algebras. If $M \in \mathbf{A}_\lambda\text{-mod}$ and $N \in \mathbf{B}_\mu\text{-mod}$, then there exists a surjective homomorphism of $\mathfrak{g} \otimes C$ -modules*

$$\tau: \mathbf{W}_C^{\lambda+\mu}(\Delta_{\lambda,\mu}^*(M \otimes N)) \twoheadrightarrow \pi_A^*(\mathbf{W}_A^\lambda M) \otimes \pi_B^*(\mathbf{W}_B^\mu N).$$

Proof. Since C is assumed to be equal to $A \oplus B$, we have that $\mathbf{U}(\mathfrak{g} \otimes C) \cong \mathbf{U}(\mathfrak{g} \otimes A) \otimes \mathbf{U}(\mathfrak{g} \otimes B)$ and that $\pi_A^* W_A(\lambda) \otimes \pi_B^* W_B(\mu)$ is a $\mathbf{U}(\mathfrak{g} \otimes C)$ -module with (see Remark 8.1(a))

$$(u_a u_b)(w_1 \otimes w_2) = (u_a w_1) \otimes (u_b w_2), \quad u_a \in \mathbf{U}(\mathfrak{g} \otimes A), \quad u_b \in \mathbf{U}(\mathfrak{g} \otimes B), \quad w_1 \in W_A(\lambda), \quad w_2 \in W_B(\mu).$$

Thus, notice that $w_\lambda \otimes w_\mu$ generates $\pi_A^* W_A(\lambda) \otimes \pi_B^* W_B(\mu)$ and satisfies (5.4). Hence, there exists a unique surjective homomorphism of $\mathbf{U}(\mathfrak{g} \otimes C)$ -modules

$$\phi: W_C(\lambda + \mu) \rightarrow \pi_A^* W_A(\lambda) \otimes \pi_B^* W_B(\mu) \quad \text{such that} \quad \phi(w_{\lambda+\mu}) = w_\lambda \otimes w_\mu.$$

Now, recall from (6.1) that $W_A(\lambda)$ (resp. $W_B(\mu)$) is a (right) \mathbf{A}_λ -module (resp. right \mathbf{B}_μ -module), with the (right) action of \mathbf{A}_λ (resp. \mathbf{B}_μ) being induced from the (left) action of $\mathbf{U}(\mathfrak{g} \otimes A)$ (resp. $\mathbf{U}(\mathfrak{g} \otimes B)$). Hence, by Remark 8.1(b), $\pi_A^* W_A(\lambda)$ is a \mathbf{C}_λ -module and $\pi_B^* W_B(\mu)$ is a \mathbf{C}_μ -module, by Remark 8.1(a), $\Delta_{\lambda,\mu}^*(\pi_A^* W_A(\lambda) \otimes \pi_B^* W_B(\mu))$ is a $\mathbf{C}_{\lambda+\mu}$ -module, and by (6.1), ϕ induces a surjective homomorphism of $\mathbf{C}_{\lambda+\mu}$ -modules (which we denote by the same symbol)

$$\phi: W_C(\lambda + \mu) \rightarrow \Delta_{\lambda,\mu}^*(\pi_A^* W_A(\lambda) \otimes \pi_B^* W_B(\mu)) \quad \text{such that} \quad \phi(w_{\lambda+\mu}) = w_\lambda \otimes w_\mu.$$

Notice that as a vector space $\pi_A^* W_A(\lambda) \otimes \pi_B^* W_B(\mu)$ is isomorphic to $\Delta_{\lambda,\mu}^*(\pi_A^* W_A(\lambda) \otimes \pi_B^* W_B(\mu))$. So, for the rest of this proof, we will abuse notation and denote by $W_A(\lambda) \otimes W_B(\mu)$ the (left) $\mathfrak{g} \otimes C$ -module $\pi_A^* W_A(\lambda) \otimes \pi_B^* W_B(\mu)$ and the (right) \mathbf{C}_λ -module $\Delta_{\lambda,\mu}^*(\pi_A^* W_A(\lambda) \otimes \pi_B^* W_B(\mu))$.

Since $-\otimes_{\mathbf{C}_{\lambda+\mu}} \Delta_{\lambda,\mu}^*(M \otimes N)$ is a right exact functor, the map

$$\phi \otimes \text{id}: W_C(\lambda + \mu) \otimes_{\mathbf{C}_{\lambda+\mu}} \Delta_{\lambda,\mu}^*(M \otimes N) \rightarrow (W_A(\lambda) \otimes W_B(\mu)) \otimes_{\mathbf{C}_{\lambda+\mu}} \Delta_{\lambda,\mu}^*(M \otimes N)$$

is in fact a surjective homomorphism of $(\mathfrak{g} \otimes C, \mathbf{C}_{\lambda+\mu})$ -bimodules.

Finally, using [Kum02, Lemma 3.1.7(2)], we see that the map

$$\begin{aligned} \psi: (W_A(\lambda) \otimes W_B(\mu)) \otimes_{\mathbf{C}_{\lambda+\mu}} \Delta_{\lambda,\mu}^*(M \otimes N) &\longrightarrow \pi_A^*(\mathbf{W}_A^\lambda M) \otimes \pi_B^*(\mathbf{W}_B^\mu N) \\ (w \otimes w') \otimes (m \otimes n) &\longmapsto (w \otimes m) \otimes (w' \otimes n) \end{aligned}$$

is in fact an isomorphism of $\mathfrak{g} \otimes C$ -modules. Thus $\tau = \psi \circ (\phi \otimes \text{id})$ is a surjective homomorphism of $\mathfrak{g} \otimes C$ -modules. \square

When M, N, A, B are finite dimensional and \mathfrak{g} is simple, the homomorphism τ given in Proposition 8.2 is in fact an isomorphism. In order to prove this fact, we will prove a finite-dimensional version of Theorem 6.9. We begin with some homological results.

Lemma 8.3. *Let $\lambda \in X^+$, $M \in \mathcal{C}_A^\lambda$ and $m \geq 0$. If $\text{Ext}_{\mathcal{C}_A^\lambda}^m(M, N) = 0$ for all finite-dimensional irreducible $N \in \mathcal{C}_A^\lambda$ satisfying $N_\lambda = 0$, then $\text{Ext}_{\mathcal{C}_A^\lambda}^m(M, N) = 0$ for all finite-dimensional $N \in \mathcal{C}_A^\lambda$ satisfying $N_\lambda = 0$.*

Proof. The proof is standard. It uses induction on the length of N , which is finite since N is assumed to be finite dimensional. \square

Lemma 8.4. *Let $\lambda \in X^+$. A finite-dimensional module $M \in \mathcal{C}_A^\lambda$ satisfies $\mathbf{W}_A^\lambda \mathbf{R}_A^\lambda M \cong M$, if and only if, for every finite-dimensional, irreducible $N \in \mathcal{C}_A^\lambda$ satisfying $N_\lambda = 0$, we have*

$$(8.1) \quad \text{Hom}_{\mathcal{C}_A^\lambda}(M, N) = \text{Ext}_{\mathcal{C}_A^\lambda}^1(M, N) = 0.$$

Proof. The *only if* part is an immediate consequence of Theorem 6.9. The proof of the *if* part is similar to the proof the *if* part of Theorem 6.9, but uses Lemma 8.3 and the finite dimensionality of M and N . \square

Theorem 8.5. *Let \mathfrak{g} be a finite-dimensional simple Lie superalgebra not of type $\tilde{S}(n)$, $\mathfrak{p}(n)$, or $\mathfrak{q}(n)$, with a fixed triangular decomposition satisfying (C2). Suppose also that A and B are finite-dimensional commutative, associative \mathbb{k} -algebras with unit and let $\pi_A: A \oplus B \rightarrow A$ and $\pi_B: A \oplus B \rightarrow B$ be the canonical projections. Let $\lambda, \mu \in X^+$ be such that $\lambda + \mu \in X^+$. If $M \in \mathbf{A}_\lambda\text{-mod}$, $N \in \mathbf{B}_\mu\text{-mod}$ are finite dimensional, then there is an isomorphism of $\mathfrak{g} \otimes (A \oplus B)$ -modules*

$$\mathbf{W}_{A \oplus B}^{\lambda+\mu}(\Delta_{\lambda,\mu}^*(M \otimes N)) \cong \pi_A^*(\mathbf{W}_A^\lambda M) \otimes \pi_B^*(\mathbf{W}_B^\mu N).$$

Proof. Recall from Remark 8.1, that as vector spaces, $\pi_A^* \mathbf{W}_A^\lambda M$ is isomorphic to $\mathbf{W}_A^\lambda M$, $\pi_B^* \mathbf{W}_B^\mu N$ is isomorphic to $\mathbf{W}_B^\mu N$, and $\Delta_{\lambda,\mu}^*(M \otimes N)$ is isomorphic to $(M \otimes N)$. So, during this proof, we will abuse notation and omit the pull-backs.

Now, notice that

$$\begin{aligned} \mathbf{R}_{A \oplus B}^{\lambda+\mu}(\mathbf{W}_A^\lambda M \otimes \mathbf{W}_B^\mu N) &\cong \sum_{\xi+\eta=\lambda+\mu} (\mathbf{W}_A^\lambda M)_\xi \otimes (\mathbf{W}_B^\mu N)_\eta \\ &\cong (\mathbf{W}_A^\lambda M)_\lambda \otimes (\mathbf{W}_B^\mu N)_\mu \\ &\cong M \otimes N. \end{aligned}$$

Thus, $\mathbf{W}_{A \oplus B}^{\lambda+\mu} \mathbf{R}_{A \oplus B}^{\lambda+\mu}(\mathbf{W}_A^\lambda M \otimes \mathbf{W}_B^\mu N) \cong \mathbf{W}_{A \oplus B}^{\lambda+\mu}(M \otimes N)$. Since M, N, A and B are assumed to be finite dimensional, by Corollary 7.7, $\mathbf{W}_A^\lambda M \otimes \mathbf{W}_B^\mu N$ is finite dimensional. Thus, by Lemmas 8.3 and 8.4, to prove that $\mathbf{W}_{A \oplus B}^{\lambda+\mu} \mathbf{R}_{A \oplus B}^{\lambda+\mu}(\mathbf{W}_A^\lambda M \otimes \mathbf{W}_B^\mu N) \cong (\mathbf{W}_A^\lambda M \otimes \mathbf{W}_B^\mu N)$ is equivalent to proving that

$$\text{Hom}_{\mathcal{C}_{A \oplus B}^{\lambda+\mu}}(\mathbf{W}_A^\lambda M \otimes \mathbf{W}_B^\mu N, U) = \text{Ext}_{\mathcal{C}_{A \oplus B}^{\lambda+\mu}}^1(\mathbf{W}_A^\lambda M \otimes \mathbf{W}_B^\mu N, U) = 0$$

for all finite-dimensional irreducible $U \in \mathcal{C}_{A \oplus B}^{\lambda+\mu}$ with $U_{\lambda+\mu} = 0$.

Let U be a finite-dimensional irreducible $U \in \mathcal{C}_{A \oplus B}^{\lambda+\mu}$. Since A and B are finite dimensional, there exist $\nu_A, \nu_B \in X^+$ such that U_A is an irreducible $\mathfrak{g} \otimes A$ -module of highest-weight ν_A , U_B is an irreducible $\mathfrak{g} \otimes B$ -module of highest-weight ν_B , and $U \cong U_A \otimes U_B$ (see [Che95, Proposition 8.4]). Moreover, since $\nu_A + \nu_B$ is in the set of weights of U , we have $\nu_A + \nu_B \leq \lambda + \mu$. Now, using the Künneth formula, we have:

$$\begin{aligned} \text{Hom}_{\mathcal{C}_{A \oplus B}^{\lambda+\mu}}(\mathbf{W}_A^\lambda M \otimes \mathbf{W}_B^\mu N, U) &\cong \text{Hom}_{\mathcal{C}_A^\lambda}(\mathbf{W}_A^\lambda M, U_A) \otimes \text{Hom}_{\mathcal{C}_B^\mu}(\mathbf{W}_B^\mu N, U_B), \\ \text{Ext}_{\mathcal{C}_{A \oplus B}^{\lambda+\mu}}^1(\mathbf{W}_A^\lambda M \otimes \mathbf{W}_B^\mu N, U) &\cong \text{Ext}_{\mathcal{C}_A^\lambda}^1(\mathbf{W}_A^\lambda M, U_A) \otimes \text{Hom}_{\mathcal{C}_B^\mu}(\mathbf{W}_B^\mu N, U_B) \\ &\quad \oplus \text{Hom}_{\mathcal{C}_A^\lambda}(\mathbf{W}_A^\lambda M, U_A) \otimes \text{Ext}_{\mathcal{C}_B^\mu}^1(\mathbf{W}_B^\mu N, U_B). \end{aligned}$$

Thus, if we prove that either $(U_A)_\lambda = 0$ or $(U_B)_\mu = 0$, by Theorem 6.9, we will have finished our proof.

First, assume that either $\nu_A \leq \lambda$ or $\nu_B \leq \mu$, and recall that $\nu_A + \nu_B \leq \lambda + \mu$ by construction. If $U_{\lambda+\mu} = 0$, then either $(U_A)_\lambda = 0$ or $(U_B)_\mu = 0$. Now, assume that $\nu_A \not\leq \lambda$ and $\nu_B \not\leq \mu$. In this case, if $U_{\lambda+\mu} = 0$ and $\lambda \leq \nu_A$, then $\lambda + \nu_B \leq \nu_A + \nu_B < \lambda + \mu$, which contradicts the fact that $\nu_B \not\leq \mu$. Thus, since $\lambda \not\leq \nu_A$, $\nu_A \not\leq \lambda$ and U_A is a highest-weight module of highest weight ν_A , we conclude that $(U_A)_\lambda = 0$. \square

9. LOCAL WEYL MODULES

In this section we will assume that \mathfrak{g} is either $\mathfrak{sl}(n, n)$ with $n \geq 2$, or a finite-dimensional simple Lie superalgebra not of type $\tilde{S}(n)$ or $\mathfrak{q}(n)$, and that A is an associative commutative finitely-generated \mathbb{k} -algebra with unit.

Definition 9.1 (Local Weyl module). Assume that $\psi \in (\mathfrak{h} \otimes A)^*$ and $\psi|_{\mathfrak{h}} \in X^+$. The *local Weyl module* $W_A(\psi)$ associated to ψ is defined to be the $\mathfrak{g} \otimes A$ -module generated by a vector w_ψ with defining relations

$$(9.1) \quad (\mathfrak{n}^+ \otimes A)w_\psi = 0, \quad hw_\psi = \psi(h)w_\psi, \quad (x_\alpha^-)^{\psi(h_\alpha)+1}w_\psi = 0, \quad \text{for all } h \in \mathfrak{h} \otimes A \text{ and } \alpha \in \Delta_+.$$

For the remainder of this section we assume that \mathfrak{g} is either isomorphic to $\mathfrak{sl}(n, n)$ with $n \geq 2$, or a finite-dimensional simple Lie superalgebra not of type $\tilde{S}(n)$, $\mathfrak{p}(n)$ or $\mathfrak{q}(n)$, with a fixed triangular decomposition satisfying (C2). Notice that, since \mathbf{A}_λ is a finitely-generated commutative algebra (see Proposition 7.6), every irreducible finite-dimensional \mathbf{A}_λ -module is one-dimensional. For $\psi \in (\mathfrak{h} \otimes A)^*$ such that $\psi|_{\mathfrak{h}} \in X^+$, let \mathbb{k}_ψ denote the one-dimensional irreducible \mathbf{A}_λ -module, where $xv = \psi(x)v$ for all $x \in \mathbf{A}_\lambda$ and $v \in \mathbb{k}_\psi$.

Remark 9.2. Recall that $W_A(\psi)$ is defined to be generated by w_ψ , that is, $W_A(\psi) = \mathbf{U}(\mathfrak{g} \otimes A)w_\psi$. Thus, since w_ψ satisfies $(\mathfrak{n}^+ \otimes A)w_\psi = 0$ and $hw_\psi = \psi(h)w_\psi$ for all $h \in \mathfrak{h} \otimes A$, we have $\mathbf{R}_A^{\psi|_{\mathfrak{h}}} W_A(\psi) = \mathbb{k}w_\psi$. Moreover, notice that $\mathbb{k}w_\psi$ is isomorphic to \mathbb{k}_ψ as a \mathbf{A}_λ -module.

The next result describes local Weyl modules as universal objects. It has been proved in [CLS, Proposition 4.13] when \mathfrak{g} is either a basic classical Lie superalgebra or $\mathfrak{sl}(n, n)$ for $n \geq 2$.

Proposition 9.3. *Let $\psi \in (\mathfrak{h} \otimes A)^*$ such that $\psi|_{\mathfrak{h}} = \lambda \in X^+$. Assume that $W \in \mathcal{C}_A^\lambda$ is a finite-dimensional $\mathfrak{g} \otimes A$ -module that is generated by a highest weight vector $w \in W$ such that $xv = \psi(x)v$, for all $x \in \mathfrak{h} \otimes A$. Then there exists a surjective homomorphism from $W_A(\psi)$ to W sending w_ψ to w . Moreover, $W_A(\psi)$ is the unique object in \mathcal{C}_A^λ with this property.*

Proof. This proof is similar to that of [CLS, Proposition 4.13]. □

The next result describes local Weyl modules via Weyl functors.

Theorem 9.4. *Assume that \mathfrak{g} is a finite-dimensional simple Lie superalgebra not isomorphic to either $\tilde{S}(n)$, $\mathfrak{p}(n)$ or $\mathfrak{q}(n)$. Let $\psi \in (\mathfrak{h} \otimes A)^*$ such that $\psi|_{\mathfrak{h}} = \lambda \in X^+$. Then $\mathbf{W}_A^\lambda \mathbb{k}_\psi \cong W_A(\psi)$.*

Proof. First recall from Remark 9.2 that $W_A(\psi) = \mathbf{U}(\mathfrak{g} \otimes A)w_\psi$ and $\mathbf{R}_A^\lambda W_A(\psi) = \mathbb{k}w_\psi$. Thus, there exists a unique homomorphism of $\mathfrak{g} \otimes A$ -modules $\epsilon_{W_A(\psi)}: \mathbf{W}_A^\lambda \mathbf{R}_A^\lambda W_A(\psi) \rightarrow W_A(\psi)$ satisfying

$$\epsilon_{W_A(\psi)}(u \otimes w_\psi) = uw_\psi \quad \text{for all } u \in \mathbf{U}(\mathfrak{g} \otimes A).$$

Moreover, $\epsilon_{W_A(\psi)}$ is surjective.

Now, notice that $\mathbf{W}_A^\lambda \mathbf{R}_A^\lambda W_A(\psi)$ is a $\mathfrak{g} \otimes A$ -module generated by the highest-weight vector $1 \otimes w_\psi$ (see Remarks 6.8 and 9.2). Moreover, by Corollary 7.7, $\mathbf{W}_A^\lambda \mathbf{R}_A^\lambda W_A(\psi)$ is finite dimensional. Thus, $1 \otimes w_\psi$ satisfies all the relations (5.4). This implies that we have a unique homomorphism of $\mathfrak{g} \otimes A$ -modules $\eta: W_A(\psi) \rightarrow \mathbf{W}_A^\lambda \mathbf{R}_A^\lambda W_A(\psi)$ satisfying $\eta(w_\psi) = 1 \otimes w_\psi$. Moreover, η is surjective, $\eta \circ \epsilon_{W_A(\psi)} = \text{id}_{W_A(\psi)}$, and $\epsilon_{W_A(\psi)} \circ \eta = \text{id}_{\mathbf{W}_A^\lambda \mathbf{R}_A^\lambda W_A(\psi)}$. The result follows. □

The next result is a direct consequence of Corollary 7.7 and Theorem 9.4. In the non-super setting, it was proved in [CP01, Theorem 1] for $A = \mathbb{k}[t^{\pm 1}]$, and in [FL04, Theorem 1], for the case where A is the algebra of functions on a complex affine variety. For the case where \mathfrak{g} is either basic classical or $\mathfrak{sl}(n, n)$ with $n \geq 2$, and A is finitely generated, it was proved in [CLS, Theorem 4.12].

Corollary 9.5. *If $\psi \in (\mathfrak{h} \otimes A)^*$ and $\psi|_{\mathfrak{h}} \in X^+$, then the local Weyl module $W_A(\psi)$ is finite dimensional.*

The next result follows directly from Theorems 8.5 and 9.4.

Corollary 9.6. *Let A and B be finite-dimensional commutative, associative \mathbb{k} -algebras with unit, $\psi \in (\mathfrak{h} \otimes A)^*$ such that $\psi|_{\mathfrak{h}} = \lambda \in X^+$ and $\varphi \in (\mathfrak{h} \otimes B)^*$ such that $\varphi|_{\mathfrak{h}} = \mu \in X^+$ and $\lambda + \mu \in X^+$. Then*

$$\mathbf{W}_{A \oplus B}^{\lambda + \mu}(\Delta_{\lambda, \mu}^*(\mathbb{k}_{\psi + \varphi})) \cong \pi_A^*(W_A(\psi)) \otimes \pi_B^*(W_B(\varphi))$$

as $\mathfrak{g} \otimes (A \oplus B)$ -modules.

The next result gives a homological characterization of local Weyl modules. In the non-super setting, it was proved for twisted map algebras in [FMS15, Lemma 7.5].

Corollary 9.7. *Let $\psi \in (\mathfrak{h} \otimes A)^*$ such that $\psi|_{\mathfrak{h}} = \lambda \in X^+$. A $\mathfrak{g} \otimes A$ -module V is isomorphic to the local Weyl module $W_A(\psi)$ if and only if it satisfies all of the following conditions:*

- (a) $V \in \mathcal{C}_A^\lambda$;
- (b) $\mathbf{R}_A^\lambda V \cong \mathbb{k}_\psi$;
- (c) $\mathrm{Hom}_{\mathcal{C}_A^\lambda}(V, U) = 0$ and $\mathrm{Ext}_{\mathcal{C}_A^\lambda}^1(V, U) = 0$, for all finite-dimensional irreducible $U \in \mathcal{C}_A^\lambda$ with $U_\lambda = 0$.

Proof. This proof is similar to that of [FMS15, Lemma 7.5]. \square

Lemma 9.8. *Let I_ψ be the sum of all ideals I of A such that $(\mathfrak{g} \otimes I)W(\psi) = 0$. Then I_ψ is a finite-codimensional ideal of A .*

Proof. By [Sav14, Proposition 8.1], all ideals of $\mathfrak{g} \otimes A$ are of the form $\mathfrak{g} \otimes I$, where I is an ideal of A . In particular, the annihilator of the action of $\mathfrak{g} \otimes A$ on $W_A(\psi)$ is of the form $\mathfrak{g} \otimes I$, for some ideal I of A . Since $W_A(\psi)$ is finite dimensional and $(\mathfrak{g} \otimes A)/(\mathfrak{g} \otimes I) \cong \mathfrak{g} \otimes A/I$, I must be a finite-codimensional ideal of A . Now the result follows from the fact that $I \subseteq I_\psi$. \square

Given an ideal I of A , we define its support to be the set $\mathrm{Supp}(I) = \{\mathfrak{m} \in \mathrm{MaxSpec}(A) \mid I \subseteq \mathfrak{m}\}$. The next result generalizes [CLS, Theorem 4.15].

Theorem 9.9. *Let $\psi, \varphi \in (\mathfrak{h} \otimes A)^*$, $\psi|_{\mathfrak{h}} = \lambda, \varphi|_{\mathfrak{h}} = \mu$, and suppose that $\lambda, \mu \in X^+$ are such that $\lambda + \mu \in X^+$. If $\mathrm{Supp}(I_\psi) \cap \mathrm{Supp}(I_\varphi) = \emptyset$, then (omitting the pull back maps) we have*

$$W_A(\psi + \varphi) \cong W_A(\psi) \otimes W_A(\varphi),$$

as $\mathfrak{g} \otimes A$ -modules.

Proof. Using the fact that $\mathrm{Supp}(I_\psi) \cap \mathrm{Supp}(I_\varphi) = \emptyset$, one can prove that the action of $\mathfrak{g} \otimes A$ on the tensor product $W_A(\psi) \otimes W_A(\varphi)$ descends to an action of $\mathfrak{g} \otimes (A/I_\psi \oplus A/I_\varphi)$ on $W_A(\psi) \otimes W_A(\varphi)$. By Lemma 9.8, both algebras A/I_ψ and A/I_φ are finite dimensional. Then, by Corollary 9.6, the result follows. \square

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